EULER: CONTINUED FRACTIONS AND DIVERGENT SERIES (AND NICHOLAS BERNOULLI)

CHRISTOPHER BALTUS

ABSTRACT. Euler developed the theory of continued fractions in the 1730's, driven in part by computational interests. It is perhaps a surprise that the association of a given series with a continued fraction first received full development in the case of the divergent series $1-x+2x^2-2\cdot 3x^3+2\cdot 3\cdot 4x^4-\ldots$. This connection was born during Euler's correspondence with Nicholas Bernoulli from 1742 to 1745. While he had already used divergent series to find values of convergent numerical series, in his letters with Bernoulli his concepts and practices are exposed, if not developed.

Euler a développé la théorie de fractions continues au cours des années 1730, conduit partiellement par les intérêts quantificatifs. C'est peut-être une surprise que l'association d'une série donnée avec une fraction continue a d'abord reçu le développement complet en cas de la série divergente $1-x+2x^2-2\cdot 3x^3+2\cdot 3\cdot 4x^4$ etc. Cette connexion est née pendant la correspondance d'Euler avec Nicholas Bernoulli à partir de 1742 à 1745. Pendant qu'Euler avait déjà utilisé la série divergente pour trouver des valeurs de série numérique convergente, dans ses lettres avec Bernoulli ses concepts et pratiques sont exposés, si non développé.

1. CONTINUED FRACTIONS IN THE 1730'S

Before Euler, continued fractions appeared in scattered examples, with the only coherent development amounting to a couple pages in *Arithmetica Infinitorum* [22], 1656, of John Wallis (1616-1703). That same work includes an intriguing example, discovered by William Brounker, later repeated in Wallis's *Algebra* [23, p 356][Latin edition 1693]. It is a continued fraction for the ratio of a square to its inscribed circle:

$$1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+etc.}}}} \tag{1}$$

Euler's first work with continued fractions, it appears, involved the separable Riccati equation. His continued fraction appeared in his letter to Christian Goldbach of 25 November

Date: June 2008.

1731. The separable Riccati equation

$$adq = q^2dp - dp \tag{2}$$

could, by setting $p = (2n+1)x^{\frac{1}{2n+1}}$ [20] [1], be transformed to

$$ady = y^2dx - x^{-\frac{4n}{2n+1}}dx. \tag{3}$$

He then found, leaving just meager hints of the trail he followed [See [19]],

$$q = -\frac{a}{p} + \cfrac{1}{\frac{-3a}{p} + \cfrac{1}{\frac{-5a}{p} + \cfrac{1}{\frac{-7a}{p} + \cfrac{1}{\text{etc. etc. etc.} + \cfrac{1}{\frac{-2(n-1)a}{p} + \frac{1}{x^{\frac{2n}{2n+1}}y}}}}}}. \tag{4}$$

This substitution holds if n is a positive integer.

The continued fraction reappeared in “De fractionibus continuis dissertatio,” [8] of 1737. The separable Riccati equation

$$adq + q^2dp = dp$$

yields $e^{\frac{2p}{a}} = 1 + \frac{2}{q-1}$, into which the continued fraction (4), with a replaced by $-a$, can be incorporated. With s set equal to $\frac{a}{2p}$, we arrive at

$$e^{\frac{1}{s}} = 1 + \cfrac{2}{2s - 1 + \cfrac{1}{6s + \cfrac{1}{10s + \cfrac{1}{14s + etc.}}}}. \tag{5}$$

(5) converges for positive s. The full continued fraction gives $e^{\frac{1}{s}}$ as a function of s written as an infinite continued fraction, the first such function representation. But it was an isolated case, a function emerging as the solution of a differential equation.

[8], of 1737, and “De fractionibus continuis observationes” of 1739 [10] constitute a continued fraction primer. Euler opened [8] by noting that continued fractions, although less used than infinite series and products, are quite well suited to approximate computation. In [8], Euler introduced continued fractions for e and related expressions by applying

repeated division to the decimal expansion until a pattern emerged. This all suggests a computational motivation for Euler's early work with continued fractions. Only then did he turn to justification: he showed that a continued fraction derived from (5), written as a ratio of series, satisfies the separable differential equation from which it originated.

Euler began [10] by developing formulas that produced a continued fraction corresponding to a given series. Following [10] and [8], the continued fraction

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e\ +\ etc.}}}}. \tag{6}$$

has *convergents*

$$\frac{1}{0},\quad \frac{A}{1},\quad \frac{B}{P},\quad \frac{C}{Q},\quad \frac{D}{R},\quad \text{etc. where}\quad A=a,\quad B=\alpha\cdot 1+bA, \tag{7}$$
$$C=\beta A+cB,\quad D=\gamma B+dC,\ \ldots,\ \text{and}$$
$$P=b,\quad Q=\beta\cdot 1+cP,\quad R=\gamma P+dQ,\ \ldots\ .$$

[The recurrence formula is found in Wallis's [22, Prop 191].]

Then any convergent can be written as a finite series, as, for example,

$$\frac{D}{R}=\frac{A}{1}+(\frac{B}{P}-\frac{A}{1})+(\frac{C}{Q}-\frac{B}{P})+(\frac{D}{R}-\frac{C}{Q})= \tag{8}$$
$$\frac{A}{1}+\frac{B\cdot 1-AP}{1\cdot P}+\frac{CP-BQ}{P\cdot Q}+\frac{DQ-CR}{R\cdot Q}=$$
$$\frac{a}{1}+\frac{\alpha}{1\cdot P}-\frac{\alpha\beta}{P\cdot Q}+\frac{\alpha\beta\gamma}{R\cdot Q}.$$

The last line of (8) follows from a property not proved in [10] but later proved by a loose induction argument in [13] and [15]. We can check directly that, for example,

$$B\cdot 1-AP=\alpha;\qquad CP-BQ=(\beta A+cB)P-B(\beta\cdot 1+cP) \tag{9}$$
$$=\beta(AP-B\cdot 1)+cBP-BcP=-\alpha\beta,\quad \text{etc.}$$

From (8) and (9), for a given series $\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+\ldots$, corresponding values (not unique) for $a, P, Q, R, \ldots, b, c, d, \ldots$ can be found, and from these $\alpha, \beta, \gamma, \ldots,$ giving the continued fraction.

Euler noted in Articles 5 and 6 of [8] that when $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ are all positive, then the first, third, fifth convergents, etc., namely,

$$a, \quad a + \cfrac{\alpha}{b + \cfrac{\beta}{c}}, \quad a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e}}}},$$

form an increasing sequence of numbers all less than the decreasing sequence of convergents of even order. This seems to have been sufficient reason for Euler to conclude "it will be possible to approach the true value of the continued fraction as closely as desired."

The first applications of Euler's new procedure was the continued fraction development of a sequence of definite integrals $\int \frac{1}{1+x^m} dx$ where the integration is understood to be from $x = 0$ to $x = 1.$ The integrand is expanded in a power series, integrated term by term, and then transformed by the new procedure into a continued fraction. A notable example is the case $m = 2,$ giving $\frac{\pi}{4}$. Where in [8] Article 4, Brouncker's continued fraction (1) for $\frac{4}{\pi}$ is found by a complicated argument attributed to Wallis, in [10] we have a direct development of the continued fraction from Leibniz's series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

This passage from an integral or ratio of integrals to a power series to a continued fraction is perhaps the most typical strategy for Euler. Giovanni Ferraro [17] sees the continued fraction link as essential to Euler's understanding of divergent series. After several examples of the integral to series to continued fraction passage, he wrote

> series were conceived as a mere combinatorial instrument relating continued fractions and integrals. It thus was of no importance if the series by themselves had a meaning as quantities (*i.e.*, they were convergent) or not. Moreover, even if the series had no sum in the ordinary sense, they could however obtain a meaning merely from the fact that they related two expressions of the same quantity. More specifically: if the quantity A was expressed by the integral I and by the fraction $F,$ and one could transform I into F by the series $S,$ then it seemed obvious that the series S also represented the quantity $A.$ [17, p 89]

2. DIVERGENT SERIES IN THE 1730'S

18th century mathematicians understood in practice the meaning of a convergent series. Leibniz, for example, wrote to Nicholas Bernoulli (1687-1759) in 1713 that a series was *advergent* $--$ his word for "convergent" $--$ if "it could be continued so that it differed from some finite real quantity by less than any given quantity." [3, p 370]

Euler used series without regard to their convergence in several papers of the 1730's. Not that he reached false conclusions, but he invoked principles which may not hold for divergent or conditionally convergent series. See[7], of 1734. In [9], of 1737, Euler found

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\ \dots\ = \sum_{m>1,n>1} \frac{1}{m^n-1} \ = 1.$$

Euler's proof began with the divergent series $x = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots$. Rearranging,

$$x = 1+[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots]+[\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\dots]+[\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\dots]+$$
$$[\frac{1}{6}+\frac{1}{36}+\dots]+[\frac{1}{7}+\frac{1}{49}+\dots]+[\frac{1}{10}+\frac{1}{100}+\dots]\dots, \text{ and so then}$$

$x = 1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{9}+\dots$. Cancellation gives $1 = \frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\ \dots$.

A rigorous proof can be given. Let S_k denote the partial sum $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots+\frac{1}{k}$, let T be the set of fractions of form $\frac{1}{m^n-1},\ m,\ n\geq 2$, and let T_k denote the sum of terms of T which are greater than or equal to $\frac{1}{k}$: $\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\dots+\frac{1}{j}$.

Then, rearranging,

$$S_k = 1+[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots]+[\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\dots]+[\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\dots]+$$
$$[\frac{1}{6}+\frac{1}{36}+\dots]+[\frac{1}{7}+\frac{1}{49}+\dots]+[\frac{1}{10}+\frac{1}{100}+\dots]\dots,$$

where the sums are finite. If the geometric series are all continued to infinity, we get:

$$S_k < 1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{9}+\dots+\frac{1}{j},$$

where $j\geq k$. Cancellation gives $T_k < 1$, so $\lim_{k\to\infty} T_k = L \leq 1$.

On the other hand, $S_k - T_k = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{9}+\dots+\frac{1}{k}$ (where, for simplicity, we suppose $\frac{1}{k}\notin T$ and that $k\neq n^q$ for any integer q greater than 1) =

$$[\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots]+[\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\dots]+[\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\dots]+\dots+\frac{1}{k}$$

(These geometric series are infinite.)

$$= S_k - 1 + [\frac{1}{2^l} + \frac{1}{2^{l+1}} + \frac{1}{2^{l+2}} + \ldots] + [\frac{1}{3^m} + \frac{1}{3^{m+1}} + \frac{1}{3^{m+2}} + \ldots] + [\frac{1}{5^n} + \frac{1}{5^{n+1}} + \frac{1}{5^{n+2}} + \ldots] + \ldots$$

where $2^l, 3^m, 5^n$, etc, are all greater than k, and l, m, n, etc. are all greater than 1.

Then $[\frac{1}{2^l} + \frac{1}{2^{l+1}} + \frac{1}{2^{l+2}} + \ldots] = \frac{1}{2^l}\frac{1}{1-\frac{1}{2}} \leq \frac{2}{2^l - 1}$ and, in general, $\frac{1}{j^m}\frac{1}{1-\frac{1}{j}} \leq \frac{2}{j^m - 1}$ when $j \geq 2$ and $m \geq 2$. Note that $\frac{1}{2^m - 1}$, $\frac{1}{3^n - 1}$, etc. will be distinct elements of T, all $\leq \frac{1}{k}$.

Let positive ϵ be given. Since $\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \ \ldots$ converges, there is k so that the sum of the elements of T which are $\leq \frac{1}{k}$ is $\leq \frac{\epsilon}{2}$. For such a k, then,

$$S_k - T_k < S_k - 1 + \frac{2}{2^l - 1} + \frac{2}{3^m - 1} + \frac{2}{5^n - 1} + \ldots < S_k - 1 + \epsilon.$$

Therefore, $T_k > 1 - \epsilon$, completing the argument that the sum is 1. [See also [2].]

3. CORRESONDENCE WITH NICHOLAS BERNOULLI, 1742 - 1745

Nicholas Bernoulli (1687 - 1759) had written to Leibniz on 7 April 1713 that the binomial expansion of $(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1}{2}\frac{3}{4}x^2 + \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + \ldots$, although divergent when $|x| > 1$, still had a value. Bernoulli said that the remainder term, giving the quantity by which a partial sum differs from the value of the entire series, was crucial. As the series above converged for $|x| < 1$, with a remainder for each partial sum, then allowing x to exceed 1 gave it an imaginary value determined by the remainder which was in this case infinite and imaginary. [3, p 370]

Bernoulli, after 1619, continued mathematics only as a side activity, taking a chair in Logic at Basel followed by a chair in Jurisprudence [4]. In the late 1730's, however, he offered a new proof, after several by Euler, that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$.

Euler, helping in the publication of Bernoulli's article, began a correspondence in 1742. Bernoulli was troubled by Euler's handling, in one of Euler's proofs of the same formula, of the infinite series and product for $\sin s$. Euler had

$$\sin s = s(1 - \frac{s^2}{\pi^2})(1 - \frac{s^2}{2^2\pi^2})(1 - \frac{s^2}{3^2\pi^2}) \cdots = s - \frac{s^3}{1{\cdot}2{\cdot}3} + \frac{s^5}{1{\cdot}2{\cdot}3{\cdot}4{\cdot}5} - \ + \ldots,$$

$-$ the two expressions shared the zeroes $-$ and then equated the coefficients of s^3.

Bernoulli located the problem in the sine series, writing to Euler, 13 July 1742, saying a demonstration was needed that "the series $s - \frac{s^3}{6} + \frac{s^5}{120} - \ + \ldots$ converges and gives the sine of the arc s no mattter what value is assigned to s."

Bernoulli brought up an argument credited to Cramer, that in working with an ellipse, say $x^2 + \frac{y^2}{b^2} = 1,$ whose points are given in terms of arclength s from (1, 0), then the ordinate of a point $-$ its *sine* $-$ is expanded as

$$s - \frac{s^3}{6b^4} + - \ldots .$$

Its roots are $kC, k = 0, \pm 1, \pm 2, \ldots$, where C is half the circumference of the ellipse. So following Euler's argument we would get the sum of the reciprocal squares to be the generally incorrect value $\frac{C^2}{6b^4}$.

Bernoulli's diagnosis was that the series was "made divergent as s grows, " and wondered whether all equations of infinite degree have imaginary roots. [18, p 683]

When Euler's letter of 1 September 1742 did not directly answer Bernoulli's objection, Bernoulli repeated the question in his next letter, of 24 October 1742. Apparently puzzled at Bernoulli's concern, Euler answered him on 10 November 1742:

> First, I cannot satisfactorily grasp the reason that you deny that the series $s - \frac{s^3}{6} + \frac{s^5}{120} -$ etc. can be regarded as equally giving the sine of arc s, or the product $s(1 - \frac{ss}{\pi\pi})(1 - \frac{ss}{4\pi\pi})$ etc. unless at the same time convergence is demonstrated. For this series $s - \frac{s^3}{6} + \frac{s^5}{120} -$ etc. is found = $\sin s$ by legitimate integration and it will certainly equal that sum whether it is convergent or divergent. [4, p 551]

Whether convergent or divergent!? So Euler had told Bernoulli that his identification of the series with the sine function did not depend on the convergence of the series. Might this have reignited Bernoulli's concern with divergent series?

That next letter was dated 6 April 1743. Bernoulli, again, emphasized the remainder. "... a divergent series continued to infinity always lacks something which gives the exact value of the quantity which is developed in the series. In this way, $\frac{1}{1-x}$ is not $= 1 + x + x^2 + \ldots + x^\infty$ but rather $\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^\infty + \frac{x^{\infty+1}}{1-x}$."

Euler answered 14 May 1743. He objected to consideration of the remainder in the case of divergent series. He made a distinction between a "numerum infinitum determinatum" and "infinitum absolutum." The example he gives illustrates the distinction:

> This would be false, $0 = 1 - 3 + 5 - 7 + 9 - \ldots \pm (2^\infty + 1)$, but with the idea of all boundaries removed, it can be affirmed without error that $0 = 1 - 3 + 5 - 7 + 9 -$ [etc.] in infinitum. In this way is the opinion more strongly confirmed that this way of treating the sum of series will

> in no case lead me into error. [Euler seems to have in mind the formula $\frac{1-x^2}{(1+x^2)^2} = 1 - 3x^2 + 5x^4 - 7x^6 + etc.$]

When Bernoulli answered Euler, on 29 November 1743, the discussion had clearly turned to divergent series whose terms grew in magnitude to infinity, but still suggested a connection to convergent series. He told Euler, for example,

> I do not believe that properties of finite algebraic equations, for example that the negative coefficient of the second term is equal to the sum of all roots etc., are properly applied to equations having terms continuing without bound, of which none is considered the last, and consequently in which equations neither the number nor sum of the roots can be conceived.

Bernoulli went on, through several examples, to point out the severe difficulties which arise from identifying a divergent series with "the quantity from which it is formed." For example,

> it is also absurd to say that the recurrent series $1 + 3 + 8 + 19 + 43 + etc.$ is equal to just its first term, that the total equals its smallest part, since it is formed from the quantity $\frac{1}{1-3+1+2}$."
>
> [$\frac{1}{1-3x+x^2+2x^3}$, according to [3, p 710]].

[Euler worked almost exclusively with alternating divergent series.]

Euler further clarified his concepts in his next letter, of 4 February 1744. He agreed that divergent series do not have sums in the usual sense, but

> if we wish to draw back from the common meaning of the word *sum*, so the sum of this series is not called 'aggregatum omnium terminorum,' but rather the value of that finite quantity from whose development the series resulted, then not only are the usual methods of summing, which involve contradictions, eliminated, but also we can explain in what way the summing of a divergent series will not lead us into error. . . . Further it is is to be noted that, just as this new notion of sum agrees with the customary if the series is convergent, so no confusion resulting from the introduction of this new idea is to be feared. This given, it will not be absurd if we seek the sum of the highly divergent series 1 - 2 + 6 - 24 + 120 - 720 + etc. [;] for I desire the value of the finite quantity from whose development this series arises, and as that quantity is transcendental, it is sufficient to assign an approximate value. I found this value or sum to be approximately = 0,40478 [too high by about 0.001], and so less than a half unit. Against this certainly no objection can be conceived by me, except that it should be demonstrated

> that the same series cannot arise from different finite expressions, but this is for me beyond any doubt.

[See [17] for an analysis of Euler's actual practice in handling divergent series. Euler was on safe ground in that a power series with a positive radius of covergence represents a unique function, but he lost that safety when he took up the factorial series.]

Bernoulli answered on 4 April 1745, more than a year later, with, for the divergent series, just a request: "I would be grateful if you would indicate to me the transcendental formula = 0,40478 from which arises the series 1 -2 + 6 - 24 + 120 -720 + etc," and a short comment: "if the same series could be developed from more than one distinct expression, . . . it would follow that that type of expression could not be called the value or sum of the divergent series, and thus my feeling would be confirmed, as I claim, that divergent series have no value."

Euler responded on 17 July 1745 with a thorough answer to Bernoulli's request, previewing his paper *De seriebus divergentibus*, [12], of 1754. [A paper of this name would, according to C. G. J. Jacobi, be read in 1746.] He associated the series with the curve $y = x - 1x^2 + 2x^3 - 6x^4 + \dots$:

> The nature of this curve can be expressed by a differential equation, . . . , $\frac{dy}{dx} = \frac{x-y}{xx}$ . . . whose integral, if e denotes the number whose logarithm = 1, will be $e^{\frac{-1}{x}} y = \int \frac{e^{\frac{-1}{x}} dx}{x}$, taken so it vanishes when I set $x = 0$.

Solving for y and letting $x = 1$, $\quad y = \int \frac{e^{1-\frac{1}{x}} dx}{x} = \int \frac{dz}{1-lz}$ when $z = e^{1-\frac{1}{x}}$ and l denotes the natural logarithm function. [Both integrals are definite integrals between limits 0 and 1 .] To approximate the integral, Euler then replaced lz by $l(1-t)$, which he expanded as a series, and then went on to develop the integral as the sum of a series, giving 0,59521. [It is difficult to account for Euler's computed value by means of his series.] Euler finished this section of his letter with a comment: "I can hardly believe that a case can be given where the same divergent series arises from the development of several different formulas."

A long postscript immediately followed Euler's letter. It allows us to date the first appearance of a continued fraction representation of a divergent series. Euler wrote:

> As I wrote this [above] about the divergent series 1 - 1 + 2 - 6 + 24 - 120 + 720 etc., I happened upon [incidi] another way of expressing the finite quantity from which it is born [nascitur].

$$1 - a + 2a^2 - 6a^3 + 24a^4 - 120a^5 + etc.$$

$$= \cfrac{1}{1+\cfrac{a}{1+\cfrac{a}{1+\cfrac{2a}{1+\cfrac{2a}{1+\cfrac{3a}{1+\cfrac{3a}{1+\cfrac{4a}{1\ etc[\text{to } 5a]}}}}}}}};$$

limits are easily assigned between which the value is contained, and can reach to the ratio of equality as closely as desired. Thus if $a = 1$, and the sought value is 1 - 1 + 2 - 6 + 24 - 120 + etc = s, then

$$\begin{array}{llllll} s < \frac{1}{1}; & s < \frac{2}{3}; & s < \frac{8}{13}; & s < \frac{44}{73}; & s < \frac{300}{501}; \ \ldots & s < \frac{22460}{3}; \\ s > \frac{1}{2}; & s > \frac{4}{7}; & s > \frac{20}{34}; & s > \frac{124}{209}; & s > \frac{920}{1546}; \ \ldots & s > \frac{78040}{130922} \text{ etc.} \end{array}$$

Thus I can place s between decimal fractions so it lies between the limits 0,5963107 and 0,5963764 of which the later is much closer to the true value than the former [The latter claim is not correct.] so it truly is about 0,5963475. [The decimal bounds Euler gives are values of the finite continued fractions taken as far as $\frac{9}{1+10}$ and $\cfrac{9}{1+\cfrac{10}{1+10}}$; the second should have 89 for its last two digits.]

I found by a more exact computation $s = 0,5963475922$.[The first six digits after the comma are correct.] My earlier method, by which I found $s = 0,59521$, was not correct or at least not as appropriate for computation. This value subtracted from 1 will be the value (as you may want to call it) of the series 1 - 2 + 6 - 24 + 120 - 720 + etc = 0,4036925, [Perhaps the 9 is a misprint?] which in my last letter was incorrectly 0, 40478. . . . . That the series

$$z - 1z^2 + 2z^3 - 6z^4 + 24z^5 - 120z^6 + 720z^7 - etc.$$

has a determined value can be demonstrated in the following way. Let the curve be conceived for whose abscissa $= x$ the ordinate is $y = \frac{1}{1-lx}$, then the area of the curve will be [by repeated integration by parts]

$$\int \frac{dx}{1-lx} = \frac{x}{1-lx} - \frac{1\cdot x}{(1-lx)^2} + \frac{1\cdot 2x}{(1-lx)^3} - \frac{1\cdot 2\cdot 3x}{(1-lx)^4} + etc.$$

which since it has a determinate value therefore also the series has a determinate value. [lx denotes the natural logarithm of x, and the integral is the definite integral between $x = 0$ and $x = 1$.]

The infinite series $1 - 1 + 2 - 6 + 24 - 120 + etc.$ is the development of a finite definite integral, evaluated at $x = 1$. The continued fraction is developed from the series. That the definite integral, or area $\int \frac{dx}{1-lx}$, directly gives the series 1 - 1 + 2 - 6 + 24 - 120 + 720 - etc. reassures Euler that the series has a value. [See [16, p 295]: "integration was interpreted geometrically as quadrature, i.e. the the result was exact insofar as it was geometrically conceived." [Giovanni Ferraro]]

Euler now had six correct digits after the comma, instead of the two correct digits given in the main body of the letter. That higher accuracy seems due to the continued fraction. In [12], De seriebus divergentibus, further work with the continued fraction would lead to the approximation 0,596 347 362 123 7, where the first nine digits after the comma are correct. That same paper includes several other approximation schemes, including the differences method that would be thoroughly developed in *Institutionum Calculi Differentialis, pars posterior* [11], Chapter 1. None gives the accuracy of the continued fraction nor gives upper and lower bounds to the value. It is the continued fraction that, in practice, Euler most trusted.

We note, as our own postscript, that the development of the continued fraction was set out in [12], of a decade later. It is, applied to polynomials, the same division process used to write a ratio of integers as a regular continued fraction.

[From Article 21] $A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - \ldots = \dfrac{1}{1+B}$, and

$$B \text{ will } = \frac{1x - 2x^2 + 6x^3 - 24x^4 + \ldots}{1 - 1x + 2x^2 - 6x^3 + 24x^4 - \ldots} = \frac{x}{1+C},$$

$$\ldots$$

Therefore $C = \dfrac{x - 4x^2 + 18x^3 - 96x^4 + \ldots}{1 - 2x + 6x^2 - 24x^3 - \ldots} = \dfrac{x}{1+D}$

from which $D = \dfrac{2x - 12x^2 + 72x^3 - 480x^4 + \ldots}{1 - 4x + 18x^2 - 96x^3 + \ldots} = \dfrac{2x}{1+E}$

Continuing, $E = \frac{2x}{1+F}$, $F = \frac{3x}{1+G}$, $G = \frac{3x}{1+H}$, $H = \frac{4x}{1+I}$, etc.

We arrive at

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{etc.}}}}}}}}$$

REFERENCES

[1] Daniel Bernoulli, Solutio problematis Riccatiani propositi in ASct. Lips. Suppl. Tom. VIII p. 73, *Acta Eruditorum*, 1725, 473-475.

[2] Lluís Bibiloni, On a series of Goldbach and Euler, *Amer. Math. Monthly* 113 No. 3, 2006, 206-220.

[3] M. Cantor, *Vorlesungen über Geschichte der Matematik*, Dritter Band. Teubner, Stuttgart, 1901; reprinted Johnson Reprint Corp., New York, 1965.

[4] Leonhard Euler, *Commercium Epistolicum, cum Johanne (I) Bernoulli et Nocolao (I) Bernoulli*, edited by E. Fellmann and G. Mikhajlov, in *Leonhardi Euleri Opera Omnia*, sub ausp. Soc. Scient, Nat. Helv. 1911- , Series 4A vol. 2, Birkhäuser, Basel, 1998.

[5] Leonhard Euler, Nova methodus innumerabiles aequationes differentiales secundi gradus reducendi ad aequationes differentiales primi gradus, *Comm. Acad. Sci. Petrop.* 3 1728 (1732) 124-137, in *Euler Opera Omnia* 1911- , Series 1 vol. 22, 1-14.

[6] Leonhard Euler, Constructio aequationis differentialis $ax^n dx = dy + y^2 dx$, *Comm. Acad. Sci. Petrop.* 6 1732/33 (1738) 124-137, in *Euler Opera Omnia* 1911- , Series 1 vol. 22, 19-35.

[7] Leonhard Euler, De progressionibus harmonicis observationes , *Comm. Acad. Sci. Petrop.* 7, 1734 (1740) 150-161 , in *Euler Opera Omnia* 1911- , Series 1 vol. 14, 87 - 100.

[8] Leonhard Euler, De fractionibus continuis dissertatio, *Comm. Acad. Sci. Petrop.* 9 1737 (1744) 98–137, in *Euler Opera Omnia* Series 1 vol. 14, 187–216; translation by M. F. Wyman and B. F. Wyman as An essay on continued fractions, *Math. Systems Theory* 18, 1985, 295-328.

[9] Leonhard Euler, Variae observationes circa series infinitas , *Comm. Acad. Sci. Petrop.* 9, 1737 (1744) 160-188 , in *Euler Opera Omnia* 1911- , Series 1 vol. 14, 217 - 244.

[10] Leonhard Euler, De fractionibus continuis observationes, *Comm. Acad. Sci. Petrop.* 11 1739 (1750) 32-81, in *Euler Opera Omnia* Series 1 vol. 14, 291 - 349.

[11] Leonhard Euler, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum* St. Petersberg, 1755, in *Euler Opera Omnia* Series 1 vol. 10

[12] Leonhard Euler, De seriebus divergentibus,, *Novi Comm. Acad. Sci. Petrop.* 5 1753 (1760) 205-237, in *Euler Opera Omnia* Series 1 vol. 14, 585 - 617.

[13] Leonhard Euler, De usu novi algorithmi in problemate Pelliano solvendo, *Novi Comm. Acad. Sci. Petrop.* 11 1765(1767) 29-66, in *Euler Opera Omnia* Series 1 vol. 3, 73-111.

[14] Leonhard Euler, *Introduction to Analysis of the Infinite,* trans. by J. D. Blandon, Springer-Verlag, New York and Berlin, 1988; from *Introductio in analysin infinitorum,* Vol 1, 1748, in *Euler Opera Omnia* Series 1 vol. 8.

[15] Leonhard Euler, De transformatione serierum in fractiones continuas, ubi simul haec theoria non mediocriter amplificatur, 1775 *Opuscula Analytica* 2, Acad. Sci. Petrop., St. Petersberg (1785) 138-177, in *Euler Opera Omnia*Series 1 vol. 15, 661 - 700.

[16] Giovanni Ferraro, Some aspects of Euler's theory of series: *Inexplicable* functions and the Euler-Maclaurin summation formula, *Hist. Math.* 25 No. 3, 1998, 290-317.

[17] Giovanni Ferraro, The value of an infinite sum. Some observations on the Eulerian theory of series, *Sciences et techniques en perspective* 4, 2000, 73-113.

[18] P. H. Fuss, editor, *Correspondance Mathématique et Physique de Quelques Célèbres Géomètres du XVIIIe Siècle*, Vol. 2, St. Petersburg, 1843.

[19] Carl Boehm, editor, Ubersicht über die bande 14, 15, 16, 16*, *Leonhardi Euleri Opera Omnia,* 1911- , Series 1 vol. 16.2 , IC.

[20] Jacobo Riccati, Animadversiones in aequationes differentiales secundi gradus, *Actorum Eruditorum Supplementa* VIII, 1724, 66-73.

[21] C. Edward Sandifer, *The Early Mathematical Works of Leonhard Euler*, Mathematical Association of America, Washington, DC, 2007

[22] John Wallis, *The Arithmetic of Infinitesimals*, Springer-Verlag, New York, 2004, translation and introduction by J. A. Stedall of *Arithmetica Infinitorum*, 1656, Oxford.

[23] John Wallis, *De Algebra Tractatus*, Henry Aldrich, Oxford, 1693, in John Wallis, *Opera Mathematica* II, Oxford, 1695, reprinted 1972.

DEPARTMENT OF MATHEMATICS, SUNY COLLEGE AT OSWEGO, OSWEGO, NY 13126 USA

E-mail address: baltus@oswego.edu