NOTES ON EULER'S CONTINUED FRACTIONS

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ABSTRACT. When Euler first worked with continued fractions, by 1731, the subject consisted of a few formulas, largely from Wallis, and a few particular continued fractions. Euler established ties to differential equations and infinite series, and studied a variety of special forms, establishing continued fractions as a field within mathematics. His work with the Pell equation illustrates the strength of his general approach together with its limitations. Euler's lesser interest in theory limited his achievement, where the young Lagrange quickly surpassed him.

1. INTRODUCTION

Mathematicians may be gathered into two groups: solvers of problems and builders of theory. In examining the continued fraction work of Leonhard Euler (1707 - 1783) from the 1730's, together with a 1759 paper, we put Euler with the first group. We see his brilliant exploitation of examples to arrive at general forms, the intense interest in computation, the discovery of connections between apparently distant ideas. At the same time we will see his limitations.

Before Euler, continued fractions appeared only in scattered examples, with the only coherent development amounting to a couple pages in *Arithmetica Infinitorum* [24], 1656, of John Wallis (1616-1703). That same work includes an intriguing example, discovered by William Brounker, later repeated in Wallis's *Algebra* [25, p 356][Latin edition 1693]. It is a continued fraction for the ratio of a square to its inscribed circle:

$$1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+etc.}}}}$$

2. THE RICCATI EQUATION, 1731

Euler seems to have first encountered continued fractions in working with, as he wrote, "the recently considered case of the separable Riccati equation." [11, p 58] Riccati had discussed the equation in a 1724 paper where, by substitution, he reduced a second order

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differential equation to first order [21]. [See [13, p 483].] Euler had worked along similar lines since 1728, when he reduced certain second order differential equations to first order by a change of variable. [3]

Euler's continued fraction result was described in his letter to Christian Goldbach of 25 November 1731, from St. Petersburg. After comments on the equation

$$dz = (p+1)zd\nu + \frac{n(1-z)d\nu}{\nu},$$

he observed that the separable Riccati equation

$$adq = q^2dp - dp \tag{1}$$

could, by setting $p = (2n+1)x^{\frac{1}{2n+1}}$, be transformed to

$$ady = y^2dx - x^{-\frac{4n}{2n+1}}dx. \tag{2}$$

The use of a power of x was made by Riccati, and Daniel Bernoulli had found the possible values for the exponent [2]. We get

$$q = -\frac{a}{p} + \cfrac{1}{\frac{-3a}{p} + \cfrac{1}{\frac{-5a}{p} + \cfrac{1}{\frac{-7a}{p} + \cfrac{1}{\text{etc. etc. etc.} + \cfrac{1}{\frac{-2(n-1)a}{p} + \frac{1}{x^{\frac{2n}{2n+1}}y}}}}}} \tag{3}$$

This substitution holds if n is a positive integer.

On the other hand, Euler continued, (2) is transformed to (1) by the corresponding substitution $x = (\frac{p}{2n+1})^{2n+1}$. And when we solve in the continued fraction (3) for $x^{\frac{2n}{2n+1}}y = y(\frac{p}{2n+1})^{2n}$, we get

$$y(\frac{p}{2n+1})^{2n} = \cfrac{1}{\frac{(2n-1)a}{p} + \cfrac{1}{\frac{(2n-3)a}{p} + \cfrac{1}{\frac{(2n-5)a}{p} + \cfrac{1}{\text{etc. etc. etc.} + \cfrac{1}{\frac{3a}{p} + \frac{1}{\frac{a}{p}+q}}}}}} \tag{4}$$

The substitution $p = (2n+1)x^{\frac{1}{2n+1}}$ is clear enough. For equation (2) suggests that y might be replaced according to

$$y = x^{-\frac{2n}{2n+1}}z. \tag{5}$$

When we, then, divide by $x^{-\frac{2n}{2n+1}}$, we arrive at

$$adz - \frac{2naz}{2n+1}x^{\frac{-2n-1}{2n+1}}dx + x^{-\frac{2n}{2n+1}}dx = x^{-\frac{2n}{2n+1}}z^2dx.$$

Now set the third term $x^{-\frac{2n}{2n+1}}dx$ equal to dp. Then $p = (2n+1)x^{\frac{1}{2n+1}}$. Substitution gives

$$adz - \frac{2nazdp}{p} + dp = z^2dp. \tag{6}$$

So far there is nothing surprising. But where does the continued fraction come from? Euler said no more about it here, in 1731, nor in his 1737 paper [6] "De fractionibus continuis dissertatio," in which he returned to this continued fraction, nor did Goldbach bring up the question of a derivation in his answer to Euler in December 1731. It is worth examining. [See Euler Vol 16.2 commentary, [20].]

After the fact – seeing the continued fraction – we make the replacement

$$\frac{1}{z} = w + \frac{(2n-1)a}{p}. \tag{7}$$

We divide each term of (6) by z^2, giving

$$\frac{a}{z^2}dz - \frac{2nadp}{zp} + \frac{dp}{z^2} = dp,$$

and then replace $\frac{1}{z}$ according to (7). Cleaning up the result gives

$$adw - \frac{2(n-1)awdp}{p} + dp = w^2dp.$$

This has the form of (6) except that n has been replaced by $n-1$. The procedure can be repeated, until the sequence $n, n-1, n-2, \ldots$ reaches 0, giving the separable equation (1).

But how did Euler think of the substitution (7)? Nothing like it had been done before. This author can only attribute it to Euler's fertile mathematical imagination.

It should be noted that in the early 1730's Euler did nothing further with this continued fraction. It appears to not have excited his interest. His long and detailed treatment of the Riccati equation, from 1732, in [5] [see also [22, p 114-121]] is by an elaborate sequence of substitutions with no hint of a continued fraction.

3. DE FRACTIONIBUS CONTINUIS DISSERTATIO, 1737

In 1737, Euler wrote the most complete treatment of continued fractions to that time. It was called "De fractionibus continuis dissertatio." [6] [In Article 2, Euler tells us that he had worked with continued fractions for a long time – despite their limited appearance in his publications.] Euler opened the work by noting that continued fractions, although less used than infinite series and products, are quite well suited to approximate computation. After giving Brouncker's continued fraction for $\frac{4}{\pi}$, he presented basic concepts and formulas of continued fractions. In particular, the continued fraction

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e\ +\ etc.}}}} \tag{8}$$

is computed by the sequence of *convergents* [modern term] forming the middle line of this table

$$\begin{array}{ccccc} a & b & c & d & \\ \dfrac{1}{0} & \dfrac{a}{1} & \dfrac{ab+\alpha}{b} & \dfrac{abc+\alpha c+\beta a}{bc+\beta} & etc. \\ \alpha & \beta & \gamma & \delta & \end{array}$$

After the first, the convergents are the sequence of truncated continued fractions, and where each numerator [denominator] is formed of the previous numerator [denominator] multiplied by the Latin letter above added to the numerator [denominator] before that multiplied by the Greek letter below. [This is essentially found in Wallis's [24, Prop 191].]

Then [Article 8] "the sum of a and all the differences [of the consecutive convergents] will be the true value of the proposed continued fraction" [Wyman translation]. This agrees with the modern definition of the value of a continued fraction, although we should recall that Euler worked without an explicit definition of the value of an infinite series. This correspondence of series and continued fractions was instrumental for Euler in finding continued fraction representations of functions and the crucial link of continued fractions with divergent series.

Articles 14 through 17 are devoted to computing rational approximations to π and the number of days in the year – further suggesting a computational motivation for this excursion into continued fractions. Euler noted the property of *regular* continued fractions – where $\alpha, \beta, \gamma,$ etc. [*partial numerators*] are all 1 and $a, b, c, d,$ etc. [*partial denominators*] are all positive integers – that any convergent approximates the value of the continued fraction closer than any other fraction with a smaller denominator. When (8) is regular, we have have the fortunate notation $[a, b, c, d, \ldots]$.

Articles 18 through 20 treat continued fractions for square roots of integers, a key ingredient in the continued fraction solution of the Pell equation. For a modern reader who has seen the continued fraction treatment of the Pell equation, Euler's approach is a surprise. To develop the regular continued fraction for $\sqrt{2}$, Euler began with the decimal expansion of $\sqrt{2}$, and then by "continued division" we find that, after $a = 1$, all partial denominators are, apparently, 2. In the same way, for $\sqrt{3}$, we find $a = 1$, and then the partial denominators alternate, with $b = 1, c = 2, d = 1$, etc. i.e., $\sqrt{3} = [1, 1, 2, 1, 2, \ldots]$.

Following Euler's typical approach, he immediately generalized based on these two examples. If x is the value of the continued fraction

$$a + \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b\ +\ etc.}}} \tag{9}$$

then

$$x - a = \cfrac{1}{b + \cfrac{1}{b + \cfrac{1}{b\ +\ etc.}}} = \frac{1}{b + x - a},$$

from which it follows that

$$x = a - \frac{b}{2} + \sqrt{1 + \frac{b^2}{4}}.$$

From which it immediately follows – typical of Euler – that when in (9) $b = 2a$ then the value of the continued fraction is $\sqrt{a^2 + 1}$.

A corresponding result is found for regular continued fractions whose partial denominators alternate b, c, b, c, b, etc.

The rest of [6] is devoted to continued fractions for e, the number whose natural logarithm is 1, and related expressions such as $\sqrt{e}$ and $\frac{e+1}{e-1}$. As with $\sqrt{2}$, he began by applying repeated division to the decimal expansion until a pattern emerged. But then he turned to a general derivation and proof. He returned to the Riccati equation (1) and (2), with an insignificant change in the sign of a, and the continued fraction solution (3). The solution of the separable Riccati equation

$$adq + q^2dp = dp$$

yields $e^{\frac{2p}{a}} = 1 + \frac{2}{q-1}$ into which the continued fraction (3), with a replaced by $-a$, can be incorporated. If, then, "n is assumed to be an infinite number," we have an infinite continued fraction. From this we can get, for example, $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$. As before, typically, Euler turned to a generalization of regular continued fractions whose denominators form "an interrupted arithmetic progression." He, finally, showed that the continued fraction satisfies the separable differential equation from which it originated. He was able to write the n^{th} convergent as a ratio of series, making the value of the continued fraction, s, the ratio of the infinite series, and in a formidable display of substitutions he showed that s satisfies the Riccati equation.

4. THE PELLEQUATION, 1732

We turn now to the Pell equation, where Euler also encountered continued fractions. His first work was [4] "De solutione problematum Diophantorum per numeros integros," of 1732. The continued fraction here was implicit, not explicit, further suggesting Euler's lack of interest during the early 1730's. The Diophantine problem was to find integer values for x which make a given polynomial $ax^2 + bx + c$ a perfect square. Given one solution and a solution to the Pell equation $ap^2 + 1 = q^2$, then an infinite sequence of solutions can be produced. Euler provided a procedure to solve the Pell equation and claimed, correctly but without proof, that it always had solutions.

Euler set out his method of solution in the example $31p^2+1 = q^2$, following Wallis ina procedure attributed to Brouncker [23, Wallis to Brouncker, 17 Dec 1657, p 789-797] and in [25, Chapter 98]. As q lies between $5p$ and $6p$, then we let $q = 5p + a$. We substitute to reach a new equation and then repeat the process. The steps are set out in a table:

$$\begin{array}{lcr}
 & & q = 5p + a \\
6p^2 = 10ap + a^2 - 1 & p = \dfrac{5a + \sqrt{31a^2 - 6}}{6} & p = a + b \\
5a^2 = 2ab + 6b^2 + 1 & a = \dfrac{b + \sqrt{31b^2 + 5}}{5} & a = b + c \\
3b^2 = 8bc + 5c^2 - 1 & b = \dfrac{4c + \sqrt{31c^2 - 3}}{3} & b = 3c + d \\
2c^2 = 10cd + 3d^2 + 1 & c = \dfrac{5d + \sqrt{31d^2 + 2}}{2} & c = 5d + e \\
3d^2 = 10de + 2e^2 - 1 & d = \dfrac{5e + \sqrt{31e^2 - 3}}{3} & d = 3e + f \\
5e^2 = 8ef + 3f^2 + 1 & e = \dfrac{4f + \sqrt{31f^2 + 5}}{5} & e = 2f + g \\
f^2 = 12fg - 5g^2 + 1 & f = 56g + \sqrt{31g^2 + 1}. &
\end{array}$$

We can now let $g = 0$ and work back to find $p = 273$ and $q = 1520$.

Sequences of integers from the table would reappear in Euler's 1759 paper [8], with denominators 6, 5, 3, 2, . . . as $E_2, E_3, E_4, E_5, \ldots$.

The column on the right produces the continued fraction

$$\frac{q}{p} = 5 + \frac{a}{p} = 5 + \frac{1}{\frac{p}{a}} =$$

$$5 + \frac{1}{1 + \frac{b}{a}} = \ldots =$$

$$5 + \frac{1}{1+} \quad \frac{1}{1+} \quad \frac{1}{3+} \quad \frac{1}{5+} \quad \frac{1}{3+} \quad \frac{1}{2 + \frac{g}{f}}.$$

With $g = 0$ this is the *regular* or *simple* continued fraction expansion of $\frac{1520}{273}$ and the beginning – if the 2 is replaced by $1 + \frac{1}{1}$ – of the regular continued fraction expansion of $\sqrt{31}$.

We have here a suggestion of an opportunity missed in 1732. Where Euler has

$$e = 2f + g,$$

which allowed Wallis and Euler to terminate the continued fraction with $g = 0$, a continuation of the pattern of computations would give

$$e = f + g, \quad \text{then} \quad 6f^2 = 2fg + 5g^2 - 1 \quad \text{and} \quad f = \frac{g + \sqrt{31g^2 - 6}}{6}.$$

The right column in the table would then have

$$f = g + h \quad \text{and} \quad g = 10h + i,$$

after which the pattern would repeat indefinitely. The analysis of 1759 might have appeared 27 years earlier.

It appears that in 1732 Euler, like wallis earlier, did not recognize the continued fraction connection.

5. Square Root Continued Fractions

Euler's development of continued fractions for square roots, in [6] and [9], suggests that he may have missed the connection to the Pell equation because he gave did not need to produce the regular continued fraction expansion. In *Introduction to Analysis of the Infinite* [9] Article 377, following the method of [6], Euler showed that x, the value of the continued fraction 9 with $a = 0$, $[0, b, b, b, \ldots]$, satisfies $x^2 + bx = 1$, so $x = \frac{\sqrt{b^2+4}-b}{2}$. In the next article Euler declared that "This method does not give an approximation of the square root of all numbers." To extend the method to include all numbers, we let

$x = [0, a, b, a, b, etc.] =$

$$\frac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \dots}}}}. \tag{10}$$

This gives $ax^2 + abx = b$, so $x = \frac{\sqrt{a^2b^2+4ab}-ab}{2a}$, from which "we can obtain the square roots of all numbers." The claim, apparently, is that for all nonsquare n there are integers k, a, b so that $nk^2 = a^2b^2 + 4ab$. If $y = ab$, the equation is $y^2 + 4y - nk^2 = 0$, or $y = -2 \pm \sqrt{4 + nk^2}$. y is an integer exactly when the Pell Equation

$$w^2 = 4 + nk^2$$

has a solution. We know there is an integer solution since the Pell equation

$$np^2 + 1 = q^2$$

always has solutions. Multiplication by 4 gives

$$n(2p)^2 + 4 = (2q)^2,$$

solving the equation.

In the case of $\sqrt{31}$, we saw earlier that Euler solved $31p^2 + 1 = q^2$ with $p = 237, q = 1520$. Then $4 + nk^2 = 4 + 31(2p)^2 = 9241600 = 3040^2$. So $y = ab = -2 + 3040 = 3038 = 98 \cdot 38$. Therefore, the continued fraction $x = [0, a, b, a, b, etc.] = \frac{\sqrt{a^2b^2+4ab}-ab}{2a} = \frac{546\sqrt{31}-3038}{62}$.

Euler's square root procedure certainly lets us compute $\sqrt{n}$ as accurately as we like, for any positive integer n, but it has come with no proof that it always works. It has taken us no closer to a solution of the Pell equation. In fact, we seem to be left tracing a logical circle. We can develop a continued fraction of form $[0, a, b, a, b, \dots]$ from which $\sqrt{n}$ can be found if we first solve the Pell equation $np^2 + 1 = q^2$. As Euler soon found out, his method to solve that Pell equation was by, essentially, producing the regular continued fraction for $\sqrt{n}$.

6. The Continued Fraction Connection, 1759

Euler's breakthrough work with the Pell equation was [8] "De usu novi algorithmi in problemate Pelliano solvendo," presented to the Berlin Academy in 1759. The paper was published in the St. Petersburg Memoires for 1767, which only reached J.-L. Lagrange, apparently, at the end of 1768. The dates are significant in indicating that Lagrange's first papers on the Pell equation, [14] and [15], were written without the benefit of [8].

Euler began [8] with a review of the role of the Pell equation $pp = lqq + 1$ in finding integers x which make $lxx + mx + n$ a perfect square. He then connected that Pell equation to a continued fraction:

> since when $pp = lqq + 1$ then that makes approximately $\frac{p}{q} = \sqrt{l}$, from which it is clear that $\frac{p}{q}$ is that same fraction which so closely expresses the irrational value $\sqrt{l}$, or, which so little exceeds it that, unless larger numbers are summoned, cannot be made more accurate. Because that problem, once solved successfullly by Wallis, is the same which I have solved for some time much more easily by continued fractions. [Article 7]

Euler developed the continued fraction in examples. We will follow the modern notation of [12]. The regular continued fraction $[b_0, b_1, b_2, b_3, \ldots] =$

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 \ + \ etc.}}} \tag{11}$$

has *partial denominators* $b_0, b_1, b_2, b_3, \ldots,$ and *convergents* $\frac{A_n}{B_n}$, $n = 0, 1, 2, 3, \ldots,$ where

$$A_{-1} = 1,\ A_0 = b_0, \text{ and } \quad A_{n+1} = b_{n+1}A_n + A_{n-1}, \quad n = 0, 1, 2, 3, \ldots,$$

$$B_{-1} = 0,\ B_0 = 1, \text{ and } \quad B_{n+1} = b_{n+1}B_n + B_{n-1}, \quad n = 0, 1, 2, 3, \ldots .$$

Euler presented three sequences associated with the continued fraction: $\{b_n\}$, $\{E_n\}$, and $\{\epsilon_n\}$. [Euler used $\{v, a, b, c, \ \ldots\}$, $\{\alpha, \beta, \gamma, \ \ldots\}$, and $\{A, B, C, \ \ldots\}$ respectively.] He also denoted "the greatest integer less than or equal to " by a "$<$", for which we will use "$<*$". We follow the notation of Lagrange's [15] except that we keep b_n in place of λ_{n+1}.]

These sequences emerge naturally in the step-by-step development of the regular continued fraction for $\sqrt{n}$. We return to the example of $\sqrt{31}$, which Euler presented only briefly after more detailed work with $\sqrt{61}$ and $\sqrt{67}$. We start with

$$E_1 = 1, \text{ and } b_0 < *\sqrt{31}, \text{ and } \epsilon_1 = b_0. \tag{12}$$

So $b_0 = 5$, and then

$$\sqrt{31} = 5 + \frac{1}{x_1} \text{ where, then, } 1 < x_1 = \frac{1}{\sqrt{31} - 5} = \frac{\sqrt{31} + 5}{31 - 5^2} = \frac{\sqrt{31} + \epsilon_1}{E_2 = 31 - \epsilon_1^2}.$$

$$\text{We now let } b_1 < *x_1, \text{ or } b_1 < *\frac{\sqrt{31} + \epsilon_1}{E_2}.$$

So

$$x_1 = b_1 + \frac{1}{x_2}.$$

Then

$$\frac{1}{x_2} = \frac{\sqrt{31}+5}{6} - b_1 = \frac{\sqrt{31}+\epsilon_1}{E_2} - b_1 = \frac{\sqrt{31}-(b_1E_2-\epsilon_1)}{E_2} = \frac{\sqrt{31}-\epsilon_2}{E_2},$$

$$\text{and then} \quad 1 < x_2 = \frac{E_2}{\sqrt{31}-\epsilon_2} = \frac{E_2(\sqrt{31}+\epsilon_2)}{31-\epsilon_2^2} = \frac{\sqrt{31}+\epsilon_2}{E_3 = (31-\epsilon_2^2)/E_2}.$$

Thus we build our sequences by (12) and

$$(13) \quad E_{n+1} = (31-\epsilon_n^2)/E_n, \quad b_n <* \frac{\sqrt{31}+\epsilon_n}{E_{n+1}}, \quad \epsilon_{n+1} = b_nE_{n+1} - \epsilon_n, \text{ when } n \geq 1.$$

Euler used, as an alternative, $E_{n+1} = E_{n-1} + b_{n-1}(\epsilon_{n-1} - \epsilon_n)$, proved by (13), from which we see that every E_n is an integer.

Euler indicated bounds on these sequences in Article 13. Let b_0 be ν. First, by the middle equation of (13), $b_n \cdot E_{n+1} - \epsilon_n < \sqrt{31}$, so, still by (13), $\epsilon_n \leq \nu$ for all n. Further, b_n is less than or equal to ν when $E_{n+1} > 1$, and if we should have $E_{n+1} = 1$, then [with no proof given] $b_n = 2\nu$ and $\epsilon_n = \nu$, and then the sequence $b_1, b_2, b_3, \ldots, 120$, is produced again. In numerous examples, including the development of $\sqrt{n}$ for $n = 2, 3, \ldots$, we always reach some $E_{n+1} = 1$ and $b_n = 2\nu$, and it is assumed that $b_n = 2\nu$ will always be reached.

But there is no proof. In fact, Euler expressed his impatience. He ended Article 16:

> This notable property [that we reach a $b_n = 2\nu$ with $E_{n+1} = 1,$] which in these operations is easily seen, rather than demonstrated by a labyrinth of words, is properly seen in a large number of examples.
>
> [Hanc proprietatem insignem, quae in ipsis operationibus facilius perspicitur, quam verborum ambage demonstratur, probe notasse in sequentibus plurimum intererit.]

[Note: If $E_{n+1} = 1 = E_1$, then by (13) $\nu + \epsilon_n = b_n = \epsilon_n + \epsilon_{n+1}$ so $\epsilon_{n+1} = \nu = \epsilon_1$. We then have $E_{n+2} = 31 - \epsilon_{n+1}^2 = m - \nu^2 = E_2$, and $b_{n+1} < * \frac{\sqrt{m}+\epsilon_{n+1}}{E_{n+2}} = b_1$. Therefore, the sequences $\{E_n\}, \{\epsilon_n\}$, and $\{b_n\}$ all repeat. Moreover, when we consider, in general, an integer m not of form w^2 or w^2+1 in place of 31, we have by (13), $E_n | (\epsilon_n + \epsilon_{n-1})$, so $E_n \leq 2\nu$ and $E_n = m - \epsilon_n^2$. Now if $\epsilon_n < \nu$, then $E_n \geq m - (\nu-1)^2$, so $m - (\nu-1)^2 \leq 2\nu$. This last inequality is equivalent to $m \leq \nu^2 + 1$. This is impossible since $\nu < * \sqrt{m}$ while m does not have the form w^2 or w^2+1. So $\epsilon_n = \nu$, $E_n = m - \nu^2$, and $b_n = 2\nu$.]

Now, why does the Pell equation $p^2 - lq^2 = 1$ have a solution in positive integers for any positive nonsquare l ? Euler proved, by a loose induction argument, that when the n^{th} convergent of the regular continued fraction expansion of $\sqrt{l}$ is denoted $\frac{A_n}{B_n}$, then

$$A_n^2 - lB_n^2 = (-1)^{n+1}E_{n+2}.$$

If $E_{n+1} = 1,\ \ n > 1,$ then

$$A_{n-1}^2 - lB_{n-1}^2 = (-1)^n E_{n+1} = \pm 1.$$

With +1, we have a solution $p = A_{n-1},\ q = B_{n-1}$ to the Pell equation; in the case of -1, then $p = 2A_{n-1}^2 + 1,\ q = 2A_{n-1}B_{n-1}$ is a solution.

That $\{b_n\}$ and $\{E_n\}$ are periodic is clear enough. Lagrange gave an explicit argument in [15], pointing out that the bounded sequence of integers $\{E_n\}$ must have a repeating pair $E_n, E_{n+1},$ which by (13) determines $\epsilon_n,$ then $b_n,$ and then $\epsilon_{n+1},$ and then the entire sequence from that point forward must repeat.

The argument that we must reach an E_n equal to 1, i.e., the Pell equation has a solution, is somewhat harder. The first published proof was by Lagrange in [14] of 1768. A more concise argument, based on the equations (13), was given in Lagrange's ([15, Articles 33, 35]), written later in 1768. Those papers were written, so Lagrange wrote to Euler ([7, Letter 26, 22 Dec. 1769]), without the benefit of Euler's ([8]). [It should be pointed out that only in the crucial Article 29 did Lagrange explicitly use continued fractions in ([15]).]

Lagrange noted that, by (13),
$b_{n-1}E_n = \epsilon_{n-1} + \epsilon_n < \sqrt{31} + \epsilon_n,$ and so $E_n < \sqrt{31} + \epsilon_n.$ Now,

$$E_nE_{n+1} = 31 - \epsilon_n^2 = (\sqrt{31} - \epsilon_n)(\sqrt{31} + \epsilon_n).$$

Since $E_n < \sqrt{31} + \epsilon_n,$ then $E_{n+1} > \sqrt{31} - \epsilon_n$. Then,

$$E_{n+1}b_n = \epsilon_{n+1} + \epsilon_n > \epsilon_{n+1} + \sqrt{31} - E_{n+1},$$

$$\text{so} \quad b_n > \frac{\sqrt{31} + \epsilon_{n+1}}{E_{n+1}} - 1.$$

On the other hand, with $\epsilon_n < \sqrt{31},$ then

$$b_n = \frac{\epsilon_n + \epsilon_{n+1}}{E_{n+1}} < \frac{\sqrt{31} + \epsilon_{n+1}}{E_{n+1}}.$$

Therefore, we have the central claim

$$b_n < * \frac{\sqrt{31} + \epsilon_{n+1}}{E_{n+1}}. \tag{14}$$

Now when E_n and E_{n+1} repeat going forward, as they must, then by (13) ϵ_n repeats. But now as a consequence of (14), b_{n-1} repeats. And then, again by (13), ϵ_{n-1} repeats and then E_{n-1} repeats. Eventually we work back to E_1, which is 1. The Pell equation, $lp^2 + 1 = q^2,$ therefore, has infinitely many solutions in positive integers which appear periodically as numerators and denominators of the regular continued fraction expansion of $\sqrt{l}$.

This argument lies surely within the capacity of Euler. Mights we, then, conclude that it lay outside his interest?

Lagrange developed continued fraction solutions of equations in [16], [17], of 1769, after seeing Euler's [8], and he applied continued fractions specifically to the Pell equation $Bp^2 + A = q^2$ in [18] of 1770. However, the second part of Euler's *Algebra* [10], of 1770, "Analysis of indeterminate quantities," does not mention the continued fractions which had been so fruitful in [8]. It was only in Lagrange's *Additions*, prepared in 1770 and 1771 for the 1774 French edition of Euler's *Algebra*, that we see a full continued fraction treatment of the Pell equation, with the important theorem, in Article 38, that all solutions of $Bp^2 + A = q^2$, when B is a non-square positive integer and $0 < |A| < \sqrt{B}$, are found as numerators and denominators of convergents of the regular continued fraction development of $\sqrt{B}$. The clean and simple continued fraction development of [8] is picked up again in Legendre's *Essai sur la Théorie des Nombres* of 1798.

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