Sign-Scored Testing for Ordered Alternatives in the One-Way Layout

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Abstract

In this talk we discuss a simple, robust technique for testing in the one-way layout, where the alternative hypothesis is that the centers of the distributions in the cells are in a specified order (such as $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_c$, with at least one of the inequalities strict). We do this with linear combinations of the Mood-Brown one-sided median test statistic. The asymptotic distribution of the resulting test statistic has a simple form. The resulting test is easy to apply, shares many robustness properties with the median, and is much more powerful than methods that do not use *a priori* order information. In addition, power results enable us to compare this family of tests to other tests for ordered alternatives, to find optimal coefficients of the linear combinations for specific alternatives, and find coefficients that produce an effective omnibus test.

1 Overview

Testing and estimation in the c-sample model are generally accomplished through analysis of variance techniques; the F-test and distribution free analogues to the χ^2-test are prominent. In many applications it is reasonable to assume more about the nature of the response than inequality of parameters between treatment groups. Testing order restricted models in the c-sample problem is an extension of the one-sided test in the two-sample problem.

This work focuses on a procedure for testing the "simple order" alternative hypothesis versus the null hypothesis of homogeneity of response over the treatment groups. Assume then that the over-all sample consists of $N = \sum_{i=1}^{c} n_i$ independent random variables $X_{il} \sim F_i(\cdot)$, $i = 1, \ldots, c$, $l = 1, \ldots, n_i$, where the first subscript refers to the subsample and the second subscript indexes observations within a subsample. A proportional sampling scheme is assumed throughout; for each $i = 1, \ldots, c$, $\lim_{N \to \infty} n_i/N = \lambda_i$, where $0 < \lambda_i < 1$. We desire to test

$$\begin{aligned} H_0 &: \text{ for all } x : F_1(x) = F_2(x) = \cdots = F_c(x) \\ H_1 &: \text{ for all } x : F_1(x) \geq F_2(x) \geq \cdots \geq F_c(x) \\ &\quad\; F_i(x) > F_{i+1}(x), \text{ for some } i \text{ and some } x. \end{aligned}$$

Under H_0 the X_{il} have a common distribution function denoted by $F(\cdot)$.

Two methods for testing such hypotheses are based on

1. Bartholomew's $\overline{\chi}^2$ test and distribution-free analogues.

2. A contrast, or linear combination of two-sample, one-sided test statistics of the form

$$V(\boldsymbol{b}) = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} b_{ij} V_{ij}$$

   where V_{ij} is a two-sample statistic for testing $F_i(x) > F_j(x)$ and the b_{ij} are associated (presumably non-negative) weighting coefficients, with $\boldsymbol{b}$ denoting the $(c(c-1)/2)$-vector of b_{ij}.

Our discussion centers on tests of the contrast nature. In particular, we consider the case where V_{ij} is based on the Mood-Brown two-sample statistic. Results for V_{ij} arising from a score function such as defined by Puri [8] are available. However, these results require the score function to be everywhere differentiable, a criteria not satisfied when using a sign-scored procedure.

Denote the empirical distribution function for subsample i by $\hat{F}_i$ and let $\hat{F}_{ij}$ denote the empirical distribution function for subsamples i and j combined. Let $\hat{\eta}_{ij}$ be the sample median for samples i and j combined, uniquely defined by

$$\hat{\eta}_{ij} = \inf\{x : \hat{F}_{ij}(x) \geq 1/2\} = \hat{F}_{ij}^{-1}(1/2).$$

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Then a convenient representation for the Mood statistic for samples i and j is given by

$$M_{ij} = \#\left(X_{il} \leq \hat{\eta}_{ij}\right) = n_i \hat{F}_i\left(\hat{\eta}_{ij}\right).$$

Mood [5] is credited with the development of the distribution theory for M_{ij}. Under H_0 and assuming some mild regularity conditions, the exact distribution of M_{ij} is hypergeometric; M_{ij} is asymptotically normal:

$$\mathrm{E}\left[M_{ij}\right] \sim \frac{n_i}{2}, \quad \mathrm{Var}\left[M_{ij}\right] \sim \frac{n_i n_j}{4\left(n_i + n_j\right)}.$$

Mood [5] also developed the asymptotic distribution theory of the 2-sample statistic M_{ij} under a general hypothesis. His work was later extended by Andrews [1] in deriving limit results regarding the extension of the 2-sample Mood test to the c-sample setting, unrestricted alternative. The technique employed by both Mood and Andrews is to compute the exact joint distribution of the aggregate sample median and the number of observations from each sample left of this median, "use Stirling's formula on the factorials, take logarithms, ...," [5]. Similar, but separate developments must be taken for even and odd aggregate sample sizes. Such an approach is clearly ill-suited to development of the distribution theory of linear combinations of a number of such (dependent) statistics.

The work of Bahadur [3] on quantile representation theory post-dates that of Mood and Andrews. The approach taken here applies methods of Bahadur in developing the limiting distribution of linear combinations of 2-sample Mood statistics. The method leads to a significantly more concise treatment of the problem, applies to both even and odd sample sizes, and lends itself to an intuitive interpretation.

2 Application

If we center and scale each two-sample Mood statistic, taking $V_{ij} = \left(n_i + n_j\right)\left(M_{ij} - n_i/2\right)$ The contrast of interest is then

$$V\left(\boldsymbol{b}\right) = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} b_{ij}\left(n_i + n_j\right)\left(M_{ij} - n_i/2\right).$$

The factor $\left(n_i + n_j\right)$ serves two purposes: first to express the statistic in a form consistent with the Chernoff Savage statistics studied by Puri [8], second to allow for a very compact expression for the asymptotic variance of the statistic when all the b_{ij} are equal. There is no reason to prefer a test scaled as such over, say, one based on $\sum_{i=1}^{c-1} \sum_{j=i+1}^{c} b_{ij}\left(M_{ij} - n_i/2\right)$. If the n_i are equal the two statistics are identical, otherwise they differ and, in fact, each is superior to the other against a particular alternative. Later we show how to select the b_{ij} in a fashion that optimizes the efficiency of the procedure against a particular type of response.

In Section 3 we demonstrate the asymptotic multivariate normality of the $\left(c\left(c-1\right)/2\right)$-vector of $N^{-3/2} V_{ij}$ under the assumption that H_0 is true. Therefore the scalar random variable $N^{-3/2} V\left(\boldsymbol{b}\right)$ has a limiting normal distribution with 0 mean and variance $\tau^2 = \boldsymbol{b}^T \Sigma \boldsymbol{b}$, where the matrix Σ is described in (6). If we assume the coefficients b_{ij} are equal (without loss of generality take $b_{ij} = 1, \boldsymbol{b} = \mathbf{1}$) then $\tau^2 = \left(1 - \sum_{i=1}^{c} \lambda_i^3\right)/12$. Define $\sigma_N^2 = (N^3 - \sum_{i=1}^{c} {n_i}^3)/12$ so that $\tau/(N^{-3/2}\sigma_N) \to 1$; as a result $V(\mathbf{1})/\sigma_N \xrightarrow{\mathcal{L}} Z \sim n(0,1)$. The asymptotic size-$\alpha$ test rejects H_0 for

$$V\left(\mathbf{1}\right) > z^{\alpha}\left[\frac{N^3 - \sum_{i=1}^{c} n_i^3}{12}\right]^{1/2}. \tag{1}$$

As a special case, consider equal sample sizes (i.e., $n_i = n$, $\lambda_i = c^{-1}, i = 1, \ldots, c$, and $N = cn$) . Then (1) becomes

$$\sum_{i=1}^{c-1} \sum_{j=i+1}^{c} M_{ij} > \frac{N\left(c-1\right)}{2} + z^{\alpha}\sqrt{\frac{N\left(c^2-1\right)}{48}}.$$

For equal sample sizes within treatment groups, the test that takes $b_{ij} = 1$ for all $1 \leq i < j \leq c$ is optimally efficient (as defined in Section 2.3) against an alternative that is linear in the treatment groups.

2.1 Efficiency

Consider tests of the form $T\left(\boldsymbol{b}\right) = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} b_{ij} T_{ij}$ where T_{ij} is a competing two-sample statistic. Then the (Pitman) asymptotic relative efficiency of the test based on $V\left(\boldsymbol{b}\right)$ relative to the test based on $T\left(\boldsymbol{b}\right)$ are the same as found by Chernoff and Savage [4] for the corresponding procedures in the two-sample problem, and shown by Puri [7] to be valid also for the multi-sample problem with unrestricted alternative. For example, if $T\left(\boldsymbol{b}\right)$ is the normal theory competitor

$$T\left(\boldsymbol{b}\right) = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} b_{ij} n_i n_j \left(\overline{X}_j - \overline{X}_i\right),$$

then the asymptotic relative efficiency of $V\left(\boldsymbol{b}\right)$ relative to $T\left(\boldsymbol{b}\right)$ is $4\sigma^2 f^2\left(\eta\right)$.

2.2 Equivalence

Each test based on the normal theory statistic $T\left(\boldsymbol{b}\right) = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} b_{ij} n_i n_j (\overline{X}_j - \overline{X}_i)$ can be written in the form $T'(\boldsymbol{b}') = \sum_{i=1}^{c-1} b_i' n_i n_{i+1} (\overline{X}_{i+1} - \overline{X}_i)$, where the b_i' are functions of the b_{ij} and n_i, with $\boldsymbol{b}'$ representing the $(c-1)$-vector of b_i'. A natural question is whether a similar reduction holds for $V(\boldsymbol{b})$ when the comparisons are based on Mood's test or, for that matter, any competing two-sample procedure.

We have shown that indeed this reduction holds, that for each statistic of the form of $V(\boldsymbol{b})$ there exists an *equivalent* statistic only including comparisons between adjacent treatment groups: $V'(\boldsymbol{b}') = \sum_{i=1}^{c-1} b_i' V_{i(i+1)}$. By equivalent we mean

1. Under H_0 the appropriately normed versions have negligible difference:
$$N^{-3/2} V(\boldsymbol{b}) - N^{-3/2} V'(\boldsymbol{b}') \xrightarrow{p} 0,$$
2. The asymptotic relative efficiency of the test based on $V(\boldsymbol{b})$ relative to the test based on $V'(\boldsymbol{b}')$ is unity.

The proof of this result makes use of the structure of the rank $(c-1)$ matrix A described in (7).

One might reason that there is no *a priori* reason to favor a contrast summing over all pairs to one that sums only over adjacent pairs; this result merely confirms such a suspicion. Of course the adjacent pairs version requires less computation, and hence is preferred on this count alone.

For the equal sample sizes, equal weights case (again, optimal against a linear response), the equivalent statistic takes $b_i' = c(c-i)$. Such a statistic weights the comparisons in the center more heavily than those on the ends, thereby protecting against a huge "drop-out" in the response for the middle-most treatment groups.

2.3 Optimal Weighting Schemes

Another point favors the adjacent pairs test, namely that the components of such a contrast (the $V_{i(i+1)}$) can be weighted (the b_i') in such as fashion as to "aim" the test at a particular type of alternative.

Therefore, restrict attention to contrasts based on adjacent pair-wise statistics $V(\boldsymbol{b}) = \sum_{i=1}^{c-1} b_i V_{i(i+1)}$, where $\boldsymbol{b}$ now denotes the $(c-1)$-vector of b_i. The null distribution of $N^{-3/2} V(\boldsymbol{b})$ is asymptotically normal, $n\left(0, \boldsymbol{b}^T \Sigma \boldsymbol{b}\right)$, where Σ is now the $(c-1) \times (c-1)$ covariance matrix restricted to only adjacent comparisons:

$$\sigma_{ij} = \begin{cases} \dfrac{\lambda_i \lambda_{i+1} (\lambda_i + \lambda_{i+1})}{4} & j = i, \\ -\dfrac{\lambda_{j-1} \lambda_j \lambda_{j+1}}{4} & j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Assume a location-shift model with $\theta_i, i = 1, \ldots, c$ as the associated location parameters. Define the *relative spacings* δ_i by

$$\delta_i = \frac{\theta_{i+1} - \theta_i}{\theta_c - \theta_1}, \qquad i = 1, \ldots, c-1.$$

Under the sequence of Pitman translation alternatives used to develop efficiency results the relative spacings remain constant as $N \to \infty$. Assume then that the relative spacings are known. It is possible to select a $(c-1)$-vector of weights $\boldsymbol{b}^*$ that maximize the efficiency of the test based on the statistic $V(\boldsymbol{b})$ relative to the test $V(\boldsymbol{b}_0)$ based on a competing statistic with fixed weights $\boldsymbol{b}_0$ ($\boldsymbol{b}^*$ does not depend on $\boldsymbol{b}_0$.) We call a test based on such a collection of weights an *optimally efficient* test. Computation of the associated optimal weights is straightforward. The following result is proved in [6, pages 45–48].

Theorem 1 *Let $\boldsymbol{\gamma}$ be the $(c-1)$-vector with ith component $\lambda_i \lambda_{i+1} \delta_i$. Then $\boldsymbol{b}^* = \Sigma^- \boldsymbol{\gamma}$.*

Our results show that the test that is optimally weighted for equal relative spacings (a linear response) is generally quite robust against a violation of equal relative spacings (see [6, pages 54–64]). For example, assume that $c = 7$ and that the true relative spacings are given by $[1, 2, 4, 8, 16, 32]$. Let e_1 be the efficiency of the test optimally weighted for equal relative spacings relative to the test that is optimally weighted for the given relative spacings. Consider also e_2, the efficiency of the all-pairs equal-weights test relative to the test that is optimally weighted for the given relative spacings. In Table 1 we display these efficiencies for four sample configurations.

For example, take the case with sample proportions $\lambda_1 = 1/2$, and $\lambda_i = 1/12, i = 2, \ldots, c$, corresponding to the final row of Table 1. For "near" alternatives the test that is optimally weighted against equal relative spacings requires aproximately a 29% larger sample to achieve the same power as the test weighted for the true relative spacings (which would presumably be unknown). For the test based on all pairwise comparisons, equally weighted,

$$\sum_{i=1}^{6} \sum_{j=i+1}^{c-1} (n_i + n_j)(M_{ij} - n_i/2),$$

the required sample size is 60% larger than that needed by the test weighted for the true relative spacings.

Simulated power computations confirm these results and also indicate that the test that is optimally weighted for equal relative spacings outperforms the $\overline{\chi}^2$ analogue for many alternatives. (We would not expect such a result for alternatives "in the corners," such as spacings given by $[0, 0, 0, 0, 0, 1]$.)

Table 1: Asymptotic Relative Efficiencies

Sample Sizes (Ratios Within Groups)	Efficiency e_1	e_2
$[1:1:1:1:1:1:1]$	1.298	1.298
$[7:6:5:4:3:2:1]$	1.416	1.878
$[2:1:1:1:1:1:1]$	1.308	1.375
$[6:1:1:1:1:1:1]$	1.289	1.601

2.4 Remarks

The construction of optimally weighted tests completely generalizes to tests based on other two-sample statistics; in particular those based on other score functions, such as the Mann-Whitney and normal scores tests, and the normal theory competitor as given by $T'\left(\boldsymbol{b}'\right)$. One must be careful to express the two-sample statistic in the proper fashion (as a Chernoff-Savage statistic) in order to utilize these results directly.

3 A representation theorem for the Mood median statistic

In order to re-express the Mood statistic a result of Bahadur [3] is applied. Take H_0 to be true and assume the existence of a unique value η such that $F(\eta) = 1/2$, where F has a non-vanishing and differentiable density $f(\cdot)$ at η. Without loss of generality the result is proved for $i = 1$, $j = 2$. Proofs of the following two lemmas are provided by Serfling [9, pages 95–98].

Lemma 1 *There exists a constant $C_1 > 0$ such that*

$$|\hat{\eta}_{12} - \eta| < C_1 N^{-1/2} (\log N)^{1/2}$$

a.s. $N \to \infty$.

Lemma 2 (Bahadur [3]) *Let a_N be a sequence of positive constants $a_N \sim C_2 N^{-1/2} (\log N)^{1/2}$ where $C_2 > 0$. Take*

$$H_N = \left| \left[\hat{F}_1 (\eta + x) - \hat{F}_1 (\eta) \right] - \left[F (\eta + x) - F (\eta) \right] \right|.$$

Then

$$\sup_{|x| < a_N} H_N = R_{1N},$$

with $R_{1N} = O\left(N^{-3/4} (\log N)^{3/4}\right)$ a.s. $N \to \infty$.

A standard argument (see [6, pages 8–10]) shows that Lemmas 1 and 2 together imply the following.

$$\left[\hat{F}_1 (\hat{\eta}_{12}) - \hat{F}_1 (\eta) \right] - \left[F (\hat{\eta}_{12}) - F (\eta) \right] = R_{3N}, \qquad (2)$$

with $R_{3N} = O\left(N^{-3/4} (\log N)^{3/4}\right)$ a.s. $N \to \infty$. Similarly

$$\left[\hat{F}_2 (\hat{\eta}_{12}) - \hat{F}_2 (\eta) \right] - \left[F (\hat{\eta}_{12}) - F (\eta) \right] = R_{4N}, \qquad (3)$$

with $R_{4N} = O\left(N^{-3/4} (\log N)^{3/4}\right)$ a.s. $N \to \infty$. Subtract (3) from (2), then

$$\left[\hat{F}_1 (\hat{\eta}_{12}) - \hat{F}_1 (\eta) \right] - \left[\hat{F}_2 (\hat{\eta}_{12}) - \hat{F}_2 (\eta) \right] = R_{5N}, \qquad (4)$$

with $R_{5N} = O\left(N^{-3/4} (\log N)^{3/4}\right)$ a.s. $N \to \infty$. Note that the left-hand-side of (4) does not depend on the true, common distribution function F. By definition

$$\hat{F}_2 (\hat{\eta}_{12}) = \frac{1}{2} - \frac{n_1}{n_2} \hat{F}_1 (\hat{\eta}_{12}) + O\left(N^{-1}\right). \qquad (5)$$

Substitute the right-hand-side of (5) for $\hat{F}_2 (\hat{\eta}_{12})$ in (4) and we arrive at

$$\begin{aligned} \left[\hat{F}_1 (\hat{\eta}_{12}) - \frac{1}{2} \right] &= \frac{n_2}{n_1 + n_2} \left[\hat{F}_1 (\eta) - \frac{1}{2} \right] \\ &+ \frac{n_1}{n_1 + n_2} \left[\hat{F}_2 (\eta) - \frac{1}{2} \right] + R_{6N}, \end{aligned}$$

with $R_{6N} = O\left(N^{-3/4} (\log N)^{3/4}\right)$ a.s. $N \to \infty$.

If we take $S_i = \#\left(X_{il} \le \eta\right) = n_i \hat{F}_i (\eta)$ then S_i is the one-sample sign statistic for subsample i. Taking the general case, we have proved the following result.

Theorem 2 *Properly normed, Mood's statistic can be expressed as a difference of weighted one-sample sign statistics.*

$$\begin{aligned} \left[M_{ij} - \frac{n_i}{2} \right] &= \frac{n_j}{n_i + n_j} \left[S_i - \frac{n_i}{2} \right] \\ &- \frac{n_i}{n_i + n_j} \left[S_j - \frac{n_j}{2} \right] + R_N, \end{aligned}$$

with $R_N = O\left(N^{1/4} (\log N)^{3/4}\right)$ a.s. $N \to \infty$.

In what follows it is sufficient to take $R_N = o\left(N^{1/2}\right)$ a.s. $N \to \infty$.

4 Multivariate limiting normality

Let $d = c(c-1)/2$ and let $\boldsymbol{V}_N$ be the d-vector of centered pairwise median statistics:

$$\boldsymbol{V}_N = \begin{bmatrix} (n_1+n_2)(M_{12}-n_1/2) \\ (n_1+n_3)(M_{13}-n_1/2) \\ \vdots \\ (n_1+n_c)(M_{13}-n_1/2) \\ (n_2+n_3)(M_{13}-n_2/2) \\ \vdots \\ (n_{c-1}+n_c)\left(M_{(c-1)c}-n_{(c-1)}/2\right) \end{bmatrix}.$$

Adopt the convention of indexing the components of $\boldsymbol{V}_N$ with double subscripts, $(i,j) : 1 \le i < j \le c$. Let $\boldsymbol{S}_N$ be the c-vector of (centered and scaled) sign statistics,

$$\boldsymbol{S}_N = \begin{bmatrix} n_1^{-1/2}(S_1 - n_1/2) \\ n_2^{-1/2}(S_2 - n_2/2) \\ \vdots \\ n_c^{-1/2}(S_c - n_c/2) \end{bmatrix}.$$

Applying Theorem 2,

$$N^{-3/2}\boldsymbol{V}_N = A_N B_N \boldsymbol{S}_N + \boldsymbol{R}_N,$$

where

- A_N is a $d \times c$ matrix with components

$$(A_N)_{(ij),k} = \begin{cases} \dfrac{n_j}{N} & i = k, \\ -\dfrac{n_i}{N} & j = k, \\ 0 & \text{elsewhere.} \end{cases}$$

  The structure of the matrix A_N for the case $c = 5$ is displayed in Figure 1.

- B_N is a $c \times c$ diagonal matrix, $(B_N)_{ii} = (n_i/N)^{1/2}$.

- $\boldsymbol{R}_N$ is a d-vector of terms with (ij) term $o(1)$ a.s. $N \to \infty$.

This representation establishes the limiting null distribution of the collection of pair-wise Mood statistics.

Theorem 3 *As $N \to \infty$ the random vector $N^{-3/2}\boldsymbol{V}_N$ converges in law to a d-variate normal random vector with mean vector $\mathbf{0}$ and covariance matrix $\Sigma_{d\times d}$ given by*

$$\sigma_{(ij)(km)} = \begin{cases} \dfrac{\lambda_i\lambda_j(\lambda_i+\lambda_j)}{4} & i = j, k = m, \\ \dfrac{\lambda_i\lambda_j\lambda_m}{4} & i = k, j \neq m, \\ \dfrac{\lambda_i\lambda_j\lambda_k}{4} & i \neq k, j = m, \\ -\dfrac{\lambda_i\lambda_j\lambda_m}{4} & j = k, \\ 0 & i, j, k, m \text{ distinct.} \end{cases} \tag{6}$$

$$A_N = \begin{bmatrix} \frac{n_2}{N} & -\frac{n_1}{N} & 0 & 0 & 0 \\ \frac{n_3}{N} & 0 & -\frac{n_1}{N} & 0 & 0 \\ \frac{n_4}{N} & 0 & 0 & -\frac{n_1}{N} & 0 \\ \frac{n_5}{N} & 0 & 0 & 0 & -\frac{n_1}{N} \\ 0 & \frac{n_3}{N} & -\frac{n_2}{N} & 0 & 0 \\ 0 & \frac{n_4}{N} & 0 & -\frac{n_2}{N} & 0 \\ 0 & \frac{n_5}{N} & 0 & 0 & -\frac{n_2}{N} \\ 0 & 0 & \frac{n_4}{N} & -\frac{n_3}{N} & 0 \\ 0 & 0 & \frac{n_5}{N} & 0 & -\frac{n_3}{N} \\ 0 & 0 & 0 & \frac{n_5}{N} & -\frac{n_4}{N} \end{bmatrix}.$$

Figure 1: Structure of the A Matrix Under H_0

Proof: It is well known that the vector $\boldsymbol{S}_N$ consists of independent components and, as $n_i \to \infty$, $i = 1, \ldots, c$, $\boldsymbol{S}_N$ converges in law to a c-variate normal random vector with $\mathbf{0}$ mean and covariance matrix $(1/4)I$.

- $A_N \to A_{d\times c}$

$$(A)_{(ij),k} = \begin{cases} \lambda_j & i = k, \\ -\lambda_i & j = k, \\ 0 & \text{elsewhere.} \end{cases} \tag{7}$$

  (The structure of the matrix A is displayed for the case $c = 5$ in Figure 1, page 5. Replacing the terms n_i/N with λ_i gives A.)

- $B_N \to B_{c\times c}$ where B is diagonal, $(B)_{ii} = \lambda_i^{1/2}$.

- $\boldsymbol{R}_N \to \mathbf{0}$ a.s. $N \to \infty$.

By the multivariate version of Slutsky's Theorem,

$$N^{-3/2}\boldsymbol{V}_N = A_N B_N \boldsymbol{S}_N + \boldsymbol{R}_N \xrightarrow{\mathcal{L}} \boldsymbol{X},$$

where $\boldsymbol{X}$ is d-variate normal with mean vector $AB\mathbf{0} = \mathbf{0}$ and covariance matrix given by $\Sigma = (1/4)\,ABB^TA^T$. Multiplying the matrices yields the solution given in (6). ■

Results analogous to those of Theorems 2 and 3 exist for a general hypothesis: $X_{il} \sim F_i(\cdot)$. Slightly more stringent regularity conditions are required: It is sufficient for the densities $f_i(\cdot), i = 1, \ldots, c$ to exist and be strictly positive and differentiable on $[\eta_L, \eta_H]$ where η_i uniquely solves $F_i(\eta_i) = 1/2$, and each η_i satisfies $\eta_L \leq \eta_i \leq \eta_H$. The proof is considerably more detailed and lengthy than that demonstrated for Theorem 2; however, the flavor of the proof is as given in this paper. For details see [6, Chapter 2].

References

[1] F. C. Andrews. Asymptotic behavior of some rank tests for analysis of variance. *Annals of Mathematical Statistics*, 25:724–736, 1954.

[2] S. F. Arnold. *Mathematical Statistics*. Prentice-Hall, Englewood Cliffs, NJ, 1990.

[3] R. R. Bahadur. A note on quantiles in large samples. *Annals of Mathematical Statistics*, 37:577–580, 1966.

[4] H. Chernoff and I. R. Savage. Asymptotic normality and efficiency of certain non-parametric test statistics. *The Annals of Mathematical Statistics*, 29:972–994, 1958.

[5] A. N. Mood. On the asymptotic efficiency of certain nonparametric two-sample tests. *The Annals of Mathematical Statistics*, 25:514–522, 1954.

[6] S. R. Preston. *Sign-Scored Methods for Testing Order Restricted Models in the C-Sample Design*. PhD thesis, The Pennsylvania State University, 1993.

[7] M. L. Puri. Asymptotic efficiency of a class of c-sample tests. *The Annals of Statistics*, 35:102–121, 1964.

[8] M. L. Puri. Some distribution-free k-sample rank tests of homogeneity against ordered alternatives. *Communications on Pure and Applied Mathematics*, 18:51–63, 1965.

[9] R. J. Serfling. *Approximation Theorems of Mathematical Statistics*. Wiley, New York, NY, 1980.