Chapter 7: Superposition

• Introduction
  – The wave equation is linear, that is if \( \psi_1(x, t) \) and \( \psi_2(x, t) \) satisfy the wave equation, then so does \( \psi(x, t) = \psi_1(x, t) + \psi_2(x, t) \).
  – This suggests the "Principle of Superposition": the resultant disturbance at any point in a medium is the algebraic sum of the separate constituent waves.

• The Addition of Waves of the Same Frequency
  – A solution of the wave equation can be written as

  \[
  E(x, t) = E_0 \sin[\omega t - (kx + \epsilon)],
  \]

  in which \( E_0 \) is the amplitude of the harmonic disturbance propagating along the \( x \) axis. This is

  \[
  E(x, t) = E_0 \sin[\omega t + \alpha(x, \epsilon)].
  \]

  – Suppose there are two such waves,

  \[
  E_1 = E_{01} \sin(\omega t + \alpha_1),
  \]

  \[
  E_2 = E_{02} \sin(\omega t + \alpha_2),
  \]

  each with the same frequency and speed, coexisting in space.

  – Then \( E = E_1 + E_2 \) is such that

  \[
  E = E_0 \sin(\omega t + \alpha),
  \]

  where

  \[
  E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1),
  \]

  and

  \[
  \tan\alpha = \frac{E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2}{E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2}.
  \]

  – That is the composite wave is harmonic and of the same frequency as the constituents, although its amplitude and phase are different.

  – This phase difference between these two waves may arise from a difference in path lengths and is \( \delta \), where

  \[
  \delta = \frac{2\pi}{\lambda} (x_1 - x_2) + (\epsilon_1 - \epsilon_2).
  \]
If the wave emitters are initially in phase, then $\epsilon_1 = \epsilon_2$, and because $n = c/v = \lambda_0/\lambda$

$$\delta = \frac{2\pi}{\lambda_0} n(x_1 - x_2).$$

The quantity $n(x_1 - x_2)$ is known as the optical path difference or $OPD = \Lambda$. Thus $\delta = k_0\Lambda$.

Waves for which $\epsilon_1 - \epsilon_2$ is constant are said to be coherent.

When the $OPD$ is equal to 0, $\pm 2\pi, \pm 4\pi, ...$, the resultant amplitude is a maximum because of constructive interference. When the $OPD$ is equal to $\pm \pi, \pm 3\pi, ...$, the resultant amplitude is a minimum because of destructive interference.

- **Superposition of Many Waves**

  Consider the addition of $N$ waves such that

  $$E = \sum_{i=1}^{i=N} E_0 i \cos(\alpha_i \pm \omega t).$$

  Then

  $$E = E_0 \cos(\alpha \pm \omega t),$$

  where

  $$E_0^2 = \sum_{i=1}^{i=N} E_{0i}^2 + 2 \sum_{j \geq i}^{N} \sum_{i=1}^{N} \cos(\alpha_i - \alpha_j),$$

  and

  $$\tan \alpha = \frac{\sum_{i=1}^{i=N} E_{0i} \sin \alpha_i}{\sum_{i=1}^{i=N} E_{0i} \cos \alpha_i}.$$
• Standing Waves
  – Previously considered two harmonic waves propagating in the same direction - now consider two waves propagating in the same direction.
  – Consider \( E_l = E_0 l \sin(kx + \omega t + \epsilon_l) \) and \( E_r = E_0 r \sin(kx - \omega t + \epsilon_r) \).
  – The composite disturbance is
    \[
    E = E_l + E_r,
    \]
  and with some trigonometric manipulation,
    \[
    E(x, t) = 2E_0 l \sin(kx) \cos(\omega t),
    \]
  where without too much loss of generality, we have assumed \( E_0 l = E_0 r \).
  – This is the equation for a standing or stationary wave. Its profile does not move through space.
  – At certain points, nodes, namely \( x = 0, \lambda/2, \lambda, 3\lambda/2, \ldots \) the disturbance is zero at all times.
  – Halfway between the nodes, that is, at \( x = \lambda/4, 3\lambda/4, \ldots \), the amplitude has a maximum value of \( \pm 2E_0 l \): these are the antinodes.

• The addition of Waves of Different Frequency
  – Consider the addition of two waves travelling in the same direction but with different frequency.
  – Let \( E_1 = E_{01} \cos(k_1 x - \omega_1 t) \), and \( E_2 = E_{02} \cos(k_2 x - \omega_2 t) \), where \( k_1 > k_2 \) and \( \omega_1 > \omega_2 \).
  – The net composite wave can be formulated as
    \[
    E = 2E_{01} \cos\left(\frac{1}{2}(k_1 + k_2)x - (\omega_1 + \omega_2)t\right) \times \cos\left(\frac{1}{2}(k_1 - k_2) - (\omega_1 - \omega_2)t\right).
    \]
  – Now define \( \bar{\omega} \) and \( \bar{k} \) as the average angular frequency and average propagation number. Similarly the quantities \( \omega_m \) and \( k_m \) are the modulation frequency and modulation propagation number respectively. That is
    \[
    \bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2), \omega_m = \frac{1}{2}(\omega_1 - \omega_2),
    \]
    and
    \[
    \bar{k} = \frac{1}{2}(k_1 + k_2), k_m = \frac{1}{2}(k_1 - k_2).
    \]
  – Then
    \[
    E = 2E_{01} \cos(k_m x - \omega_m t) \cos(\bar{k} x - \bar{\omega} t).
    \]
- The total disturbance may be viewed as a travelling wave of frequency \( \bar{\omega} \) having a time-varying or modulated amplitude \( E_0(x, t) \) such that,

\[
E(x, t) = E_0(x, t)\cos(\bar{k} - \bar{\omega}t),
\]

where

\[
E_0(x, t) = 2E_0\cos(k_m x - \omega_m t).
\]

- If \( \omega_1 \approx \omega_2 \), then \( \bar{\omega} \) is much greater than \( \omega_m \) and \( E_0(x, t) \) will change slowly.

- It can be shown that

\[
E_0^2(x, t) = 2E_{01}^2[1 + \cos(2k_m x - 2\omega_m t)].
\]

- Thus \( E_0^2(x, t) \) oscillates about a value of \( 2E_{01}^2 \) with an angular frequency of \( (\omega_1 - \omega_2) \): the beat frequency.

- Thus \( E_0 \), the amplitude of the resulting wave varies at the modulation frequency but \( E_0^2 \) varies at twice that - the beat frequency.

- **Group Velocity**

- When a number of different-frequency harmonic waves superimpose to form a composite disturbance, the resulting modulation envelope will travel at a speed different from that of the constituent waves.

- Group velocity verses phase velocity.

- The phase velocity in this situation is \( v = \bar{\omega}/\bar{k} \), that is the phase velocity of the carrier wave.

- The group velocity relates to the rate at which the modulation envelope advances i.e. \( v_g = \omega_m/k_m \).

- If \( v_1 = v_2 \) then \( v_g = v \).

- Phase velocity can be greater than \( c \), but phase velocity doesn’t carry information or energy.

- Group velocity can also be greater than \( c \).

- In either case, information or energy cannot be transmitted at a speed greater than \( c \).

- **Anharmonic Periodic Waves**

- Superposition of harmonic functions having different amplitudes and frequencies produces an anharmonic wave.

- Fourier Series: A function \( f(x) \) can be synthesized by a sum of harmonic functions whose wavelengths are integral submultiples of \( \lambda \) (i.e. \( \lambda, \lambda/2, \lambda/3, \ldots \)).
– Any periodic function, \( f(x) \),

\[
   f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx
\]

where, knowing \( f(x) \),

\[
   A_m = \frac{2}{\lambda} \int_0^\lambda f(x) \cos mx \, dx,
\]

\[
   B_m = \frac{2}{\lambda} \int_0^\lambda f(x) \sin mx \, dx.
\]

– \( \omega \) is the fundamental. Subsequent omegas like \( 2\omega, 3\omega, \ldots \) are the harmonics.

– Fourier series so represented has the form

\[
   f(x) = C_0 + C_1 \cos \left( \frac{2\pi}{\lambda} x + \epsilon_1 \right) + C_2 \cos \left( \frac{2\pi}{\lambda} x + \epsilon_2 \right) + \ldots
\]

– Can extend this to non-periodic waves abd write

\[
   f(x) = \frac{1}{\pi} \left[ \int_0^\infty A(k) \cos kx \, dk + \int_0^\infty B(k) \sin kx \, dk \right],
\]

\[
   A(k) = \int_{-\infty}^\infty f(x) \cos kx \, dx,
\]

\[
   B(k) = \int_{-\infty}^\infty f(x) \sin kx \, dx.
\]

– \( A(k), B(k) \) are the contributions to angular spatial frequency between \( k, k + dk \).

– The frequency bandwidth is \( \Delta k \) or \( \Delta \omega \).