9A: $2^k \cdot 3 \cdot x^\alpha$, y^β, $2 \cdot z^\gamma$, 3^l

<u>x = 5, y = 7, z = 11:</u> β must be even, otherwise $2 \cdot 11^\gamma = 7^\beta + 1 = 8 \cdot (7^{\beta-1} - 7^{\beta-2} + ... + 1)$ would be divisible by 2^3. With $\beta = 2\beta'$, β' must also be even, otherwise $2 \cdot 11^\gamma = 7^\beta + 1 = 7^{2\beta'} + 1 = 49^{\beta'} + 1 = 50 \cdot (49^{\beta'-1} - 49^{\beta'-2} + ... + 1)$ would be divisible by 5. With $\beta' = 2\beta''$ and $\beta = 4\beta''$, β'' must be even as well, otherwise $2 \cdot 11^\gamma = 7^\beta + 1 = 7^{4\beta''} + 1 = 2{,}401^{\beta''} + 1 = 2{,}402 \cdot (2{,}401^{\beta''-1} - 2{,}401^{\beta''-2} + ... + 1)$ would be divisible by 1,201. It follows that $2^k \cdot 3 \cdot 5^\alpha = 7^\beta - 1 = 7^{8\beta'''} - 1 = (7^8)^{\beta'''} - 1 = (7^8 - 1) \cdot ((7^8)^{\beta'''-1} + ... + 1)$ is divisible by $2^6 \cdot 3 \cdot 5^2 \cdot 1{,}201$, contradiction.

<u>x = 5, y = 11, z = 7:</u> β must be even, otherwise $2 \cdot 7^\gamma = 11^\beta + 1 = 12 \cdot (11^{\beta-1} - 11^{\beta-2} + ... + 1)$ would be divisible by 3. With $\beta = 2\beta'$, β' must also be even, otherwise $2 \cdot 7^\gamma = 11^\beta + 1 = 11^{2\beta'} + 1 = 121^{\beta'} + 1 = 122 \cdot (11^{\beta'-1} - 11^{\beta'-2} + ... + 1)$ would be divisible by 61. It follows that $2^k \cdot 3 \cdot 5^\alpha = 11^\beta - 1 = 11^{4\beta''} - 1 = (11^4)^{\beta''} - 1 = (11^4 - 1) \cdot ((11^4)^{\beta''-1} + ... + 1)$ is divisible by $2^4 \cdot 3 \cdot 5 \cdot 61$, contradiction.

<u>x = 7, y = 5, z = 11:</u> $2^k \cdot 3 \cdot 7^\alpha = 2 \cdot 11^\gamma - 2 = 2 \cdot (11^\gamma - 1) = 2 \cdot 10 \cdot (11^{\gamma-1} + ... + 1)$ is divisible by 5, contradiction.

<u>x = 7, y = 11, z = 5:</u> $2^k \cdot 3 \cdot 7^\alpha = 11^\beta - 1 = 10 \cdot (11^{\beta-1} + ... + 1)$ is divisible by 5, contradiction.

<u>x = 11, y = 5, z = 7:</u> β must be even, otherwise $2 \cdot 7^\gamma = 5^\beta + 1 = 6 \cdot (5^{\beta-1} - 5^{\beta-2} + ... + 1)$ would be divisible by 3. And, with $\beta = 2\beta'$, β' must be even as well: if not then $2\cdot 7^{\gamma} = 5^{\beta} + 1 = 5^{2\beta'} + 1 = 25^{\beta'} + 1 = 26\cdot(25^{\beta'-1} - 25^{\beta'-2} + ... + 1)$ would be divisible by 13. Now the fact that β is even implies that $5^{\beta-1} + 5^{\beta-2} + 5^{\beta-3} + 5^{\beta-4} + ... + 5^3 + 5^2 + 5 + 1$ may be factored as $(5+1)\cdot(5^{\beta-2} + 5^{\beta-4} + ... + 5^2 + 1) = 6\cdot(25^{\beta'-1} + ... + 1)$. Next, $2^k\cdot 3\cdot 11^{\alpha} = 5^{\beta} - 1 = 4\cdot(5^{\beta-1} + ... + 1)$ yields $2^{k-2}\cdot 3\cdot 11^{\alpha} = 6\cdot(25^{\beta'-1} + ... + 1)$ and $2^{k-3}\cdot 11^{\alpha} = 25^{\beta'-1} + ... + 1$. Exactly as above, the fact that $\beta' = 2\beta''$ is even produces the factorization $25^{\beta'-1} + ... +1 = 26\cdot(625^{\beta''-1} + ... + 1)$, so that $2^{k-3}\cdot 11^{\alpha}$ and, of course, $2^k\cdot 3\cdot 11^{\alpha}$ is divisible by 13, contradiction. (Notice: we have not assumed $k \geq 3$!)

x = 11, y = 7, z = 5: l must be odd, otherwise $2\cdot 5^{\gamma} = 3^l - 1 = 3^{2l'} - 1 = 9^{l'} - 1 = 8\cdot(9^{l'-1} + ... + 1)$ would be divisible by 2^3. Let's set $l'' = (l-1)/2$, so that $2^k\cdot 3\cdot 11^{\alpha} = 3^l - 3 = 3\cdot(3^{l-1} - 1) = 3\cdot(9^{l''} - 1) = 3\cdot 8\cdot(9^{l''-1} + ... + 1)$ and $2^{k-3}\cdot 11^{\alpha} = 9^{l''-1} + ... + 1$. Observe that l'' is odd, otherwise $2^k\cdot 3\cdot 11^{\alpha} = 3\cdot(9^{l''} - 1) = 3\cdot(9^{2l'''} - 1) = 3\cdot(81^{l'''} - 1) = 3\cdot 80\cdot(81^{l'''-1} + ... + 1)$ would be divisible by 5. It follows that $9^{l''-1} + ... + 1$ is odd, so that $2^{k-3}\cdot 11^{\alpha} = 9^{l''-1} + ... + 1$ yields $k = 3$. But then both $2^k\cdot 3\cdot 11^{\alpha} + 2 = 2\cdot 5^{\gamma}$ and $2^k\cdot 3\cdot 11^{\alpha} - 24 = 24\cdot(11^{\alpha} - 1) = 24\cdot 10\cdot(11^{\alpha-1} + ... + 1)$ would be divisible by 5, contradiction.