first draft: summer 2000

CHAPTER 9

COLORED TILINGS

[introduction (section 9.0) *only*]

9.0 Three or more colors

9.0.1 Consistency with color generalized. Let's revisit the **pgg**-like pattern of figure 6.58, coloring it now in **three** colors, and let's look at some of this pattern's isometries' effect on color:

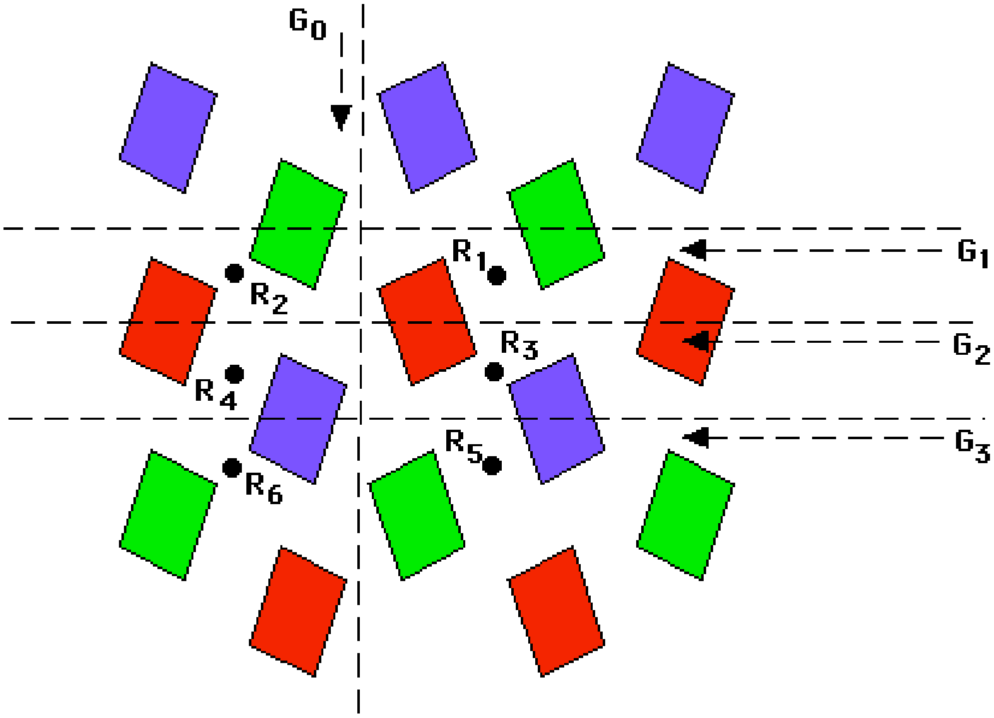

Fig. 9.1

The shown vertical glide reflection G_0 clearly takes **each** blue unit (parallelogram) to a green unit (**B** → **G**), **each** green unit to a red unit (**G** → **R**), and **each** red unit to a blue unit (**R** → **B**). Clearly all three colors are 'rotated' or 'permuted', and we denote this -- as we did back in section 7.0 without much elaboration -- using a color **permutation**, **(BGR)**: **B** goes to **G**, **G** goes to **R**, and **R** 'returns' to **B**.

The top horizontal glide reflection (G_1) certainly takes **each** blue unit to a red unit (**B** → **R**), **each** red unit to a blue unit (**R** → **B**), and **each** green unit to **another** green unit (**G** → **G**). Since G_1 swaps two colors (**B**, **R**), leaving the third (**G**) unchanged, it is reasonable to denote its effect on color by another, shorter, permutation, **(BR)**; that is, we 'record' **only** those colors that are mapped to another color, **omitting** those mapped to themselves. Likewise, the effect on color of G_2 and G_3 is denoted by **(BG)** and **(RG)**, respectively.

Looking back at G_1 above, you may very well ask: shouldn't we view it as inconsistent with color, since some colors (**B**, **R**) it 'reverses' and others (**G**) it 'preserves'? The answer is "no": what really makes an isometry **consistent with color** is its mapping of **all** units of color **X** to units of a **single** color **Y**, where **X** and **Y** may or may not be one and the same. Of course, in the **special case** of a consistent coloring involving only **two** colors, either all **X**s go to **Y**s and, **by necessity**, vice versa ("colors reversed"), or all **X**s go to **X**s and, by necessity again, all **Y**s go to **Y**s ("colors preserved"). In other words, there is no discrepancy between our definitions of "consistency with color" here and in chapters 5 and 6.

You should have no trouble verifying that the effect on color of the indicated 180^0 rotations R_1, R_2, R_3, R_4, R_5, and R_6 is **(RG)**, **(RG)**, **(BR)**, **(BR)**, **(BG)**, and **(BG)**, respectively: just as in the case of only two colors, different rotation centers may certainly have exactly the same effect on color; nor should you be surprised by the fact that **no** 180^0 rotation may rotate **three** colors (for about the same reasons that no 120^0 rotation may rotate two colors (6.13.1), actually).

How about translations? There exist vertical downward translations by one unit (**BRG**), two units (**BGR**). or three units (**P**): since in the latter case all colors are **preserved**, we return to the notation of chapter 5 and use **P** rather than an 'empty' permutation. (Likewise, **I** still stands for "inconsistent with color".)

9.0.2 <u>More than three colors.</u> There is no upper limit on the number of colors in which a wallpaper pattern may be colored, but we will limit ourselves to nine at most. In addition to the three colors already used, we may in addition use Yellow (**Y**), Azur (**A**), Violet (**V**), Orange (**O**), Khaki (**K**), and Salmon (**S**). As an example, here are two distinct colorings of the pattern from figure 6.58 in **six** colors:

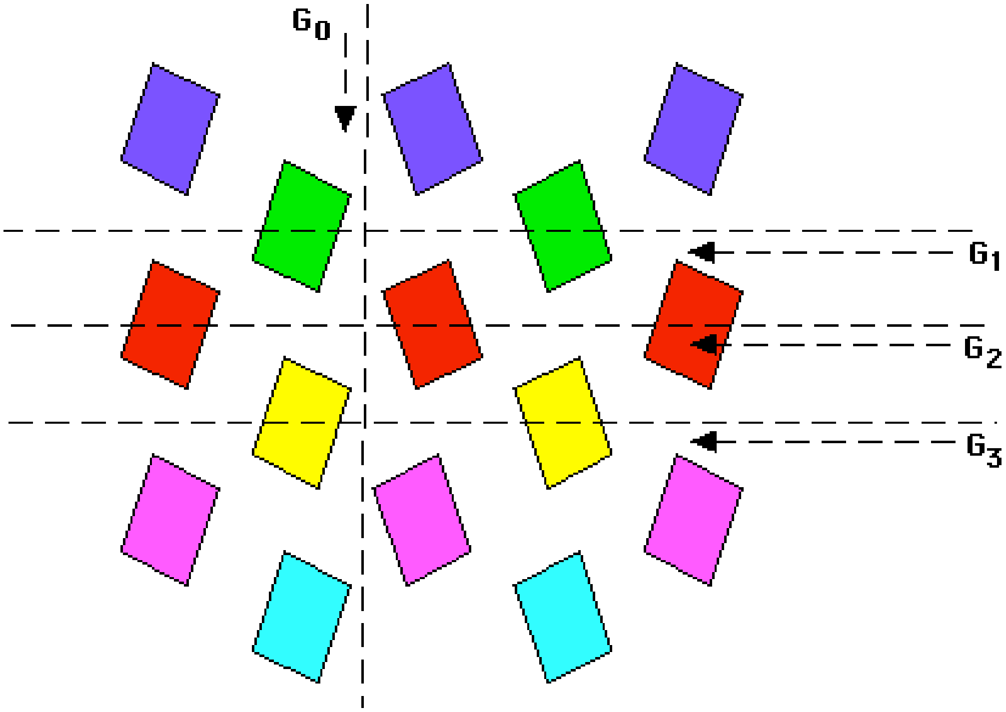

Fig. 9.2

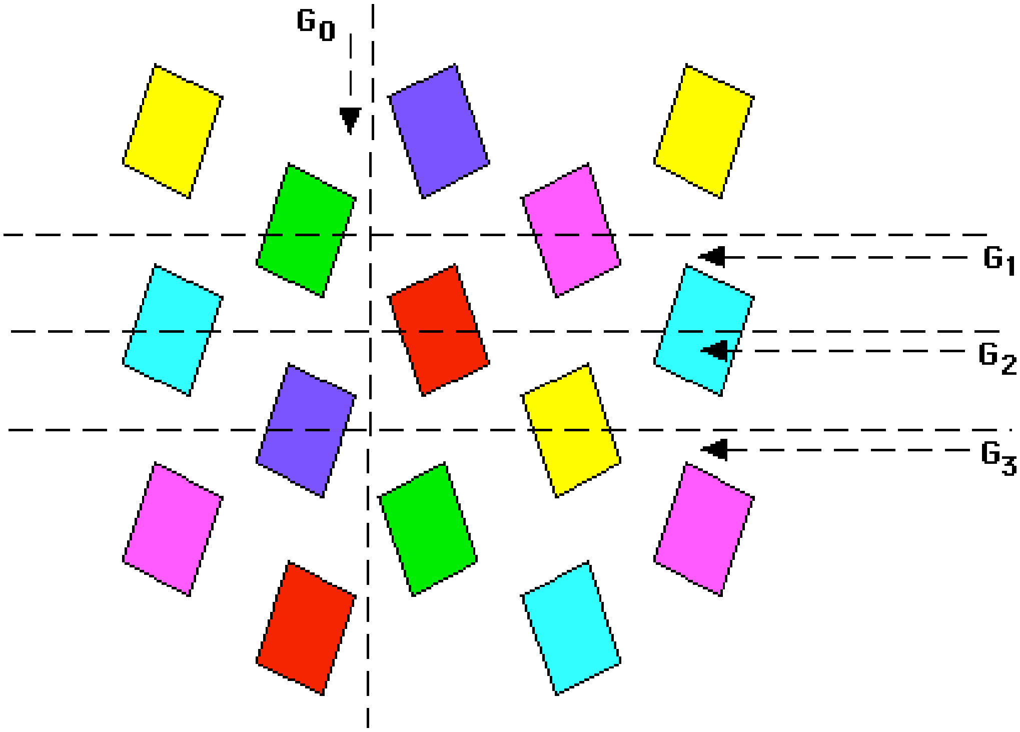

Fig. 9.3

Let's look at the color effect of G_0. In the case of figure 9.2 it's **(ABGRYV)**, that is **B → G → R → Y → V → A → B** : all six colors are rotated in a **6-cycle**. In the case of figure 9.3 it's **(AYV)(BGR)**, as all six colors have again 'moved', except this time they did so in two **3-cycles**: **B → G → R → B** and **Y → V → A → Y** . Make sure you understand the difference betweeen these two color permutations!

Let's now look at the color effect of G_2: in the case of figure 9.2 it's **(BV)(GY)**, a 'product' of two **2-cycles** that leaves **A** and **R** unchanged, while in the case of figure 9.3 it's **(AR)(BV)(GY)**, a 'product' of three 2-cycles that leaves no color unchanged. Likewise, the color effects of G_1 and G_3 in figure 9.2 are **(AY)(BR)** and **(AG)(RV)**, respectively, while in figure 9.3 they are **(AB)(GV)(RY)** and **(AG)(BY)(RV)**, respectively; you may of course have to 'extend' the patterns a bit in order to fully verify these color effects! Similar observations may be made about the pattern's other isometries' effect on color.

The two colorings provided in figures 9.2 and 9.3 are indeed **distinct** (mathematically): their vertical glide reflections differ 'structurally' (6-cycles versus products of two 3-cycles), and so do their horizontal glide reflections (products of two 2-cycles versus products of three 2-cycles). In other words, what really matters is the **structure** of the color permutations associated with the pattern's isometries; the colors themselves are not relevant at all, save perhaps for aesthetic considerations. (For example, using the same six colors always, there exist another thirty nine colorings identical mathematically to the coloring in figure 9.2!)

9.0.3 <u>How many colorings?</u> How many **distinct** color-consistent colorings of the **pgg** pattern in question do exist? **Infinitely** many! So we better ramify the question a bit, like: how many consistent colorings in **six** colors are there? You could sooner or later discover the correct answer by way of ruthless experimentation, but you don't have to: much easier would be to turn to page 250 of Thomas W. Wieting's monograph ***The Mathematical Theory of Chromatic Plane Ornaments*** (Marcel Dekker, 1982), where we see that our **pgg** pattern -- and every **pgg** pattern for that matter -- can be consistently colored employing six colors in precisely **four** ways! And on the same page we find out that the **3-coloring** of figure 9.1 is the **only** consistent one, while there exist **nine** consistent **12-colorings**, **two** consistent **9-colorings**, and so on! Where do all these numbers and coloring possibilities come from?

Unfortunately, Wieting's book is as sophisticated as colorless, so you may get a better introduction by reading Arthur L. Loeb's ***Color and Symmetry*** (Wiley, 1971). Still, the right way to attack the question of **k-colorings** of wallpaper patterns in full generality is Wieting's method, based on the mathematical theory of **Groups**. Since Group Theory is beyond the scope of this chapter and book, we limit the discussion to an incomplete and **informal**, if not 'playful', investigation. But recall that we have thoroughly answered this question for **k = 2** in chapter 6 by way of rather 'elementary' means, closer in spirit to those of Loeb's.

9.0.4 Tilings. Some of our examples of wallpaper patterns in chapters 4 and 6 do **tile** the plane: that is, there exist neither overlapping motifs (**tiles**) nor gaps between them. Such wallpaper patterns are known as **tilings** of the plane: a term fully consistent with bathroom wall etc tilings as you know them from your daily life. For example, the beehive is a **p6m** tiling of the plane, while none of the **p6m**-like two-colored patterns presented in section 6.17 is. With the single exception of figure 9.7, all tilings we will study and color in this chapter belong to one of the seventeen types of wallpaper patterns.

We prefer to study **color symmetry** in the context of tilings rather than that of non-tiling patterns (such as the one in figure 9.1), both because the effect of isometries on color is easier to decide in the case of tilings and because tilings are indeed, and have **historically** been, very much worth studying for their own sake. In fact you will learn quite a bit about tilings while ostensibly exploring various colorings; but if you really want to get into tilings, then the real source, advanced as it happens to be, is Branko Grundbaum & G. C. Shephard's classic ***Tilings and Patterns*** (Freeman, 1987).

9.0.5 Maplike colorings. As you know from your tender years, all maps depict states or countries sharing a border (consisting of more than a single point) in different colors. For example, Arizona may be of the samer color as Colorado, but it cannot be of the same color as

Utah. Departing from this fundamental map property, we call the coloring of a tiling **maplike** if and only if every two tiles (‘countries’) with common boundary (‘border’) are of different colors.

One of the most celebrated results in Mathematics, conjectured in 1852 and proven only in 1976, the **Four-Color Theorem**, states that every ‘map’ (on the plane) may indeed be colored maplike using **at most four colors**. This seemingly ‘obvious’ result -- you will certainly succeed in coloring any **specific** map in four colors, after all -- could not be mathematically proven until **computer power** became available to mathematicians. It was in fact the first time a mathematical proof had to rely on zillions of computer calculations, and that caused a sensation; a quarter of century later, computer-assisted proofs go completely unnoticed. Strangely, it was much easier to show, for example, that the **maximum** number of colors needed for a map drawn on a **torus** (doughnut) is **seven**; and similar **chromatic numbers** were relatively easily, and certainly before 1976, available for all **topological surfaces** save for our ‘simple’ plane! [For further details you may like to consult Thomas L. Saaty & Paul C. Kainen’s ***The Four-Color Problem: Assaults and Conquest*** (Dover, 1986).]

9.0.6 Faithful colorings. When a tiling is not colored maplike, it is also likely to be ‘inconsistently colored’. Compare for example the following two ‘chessboard’ colorings of the bathroom wall:

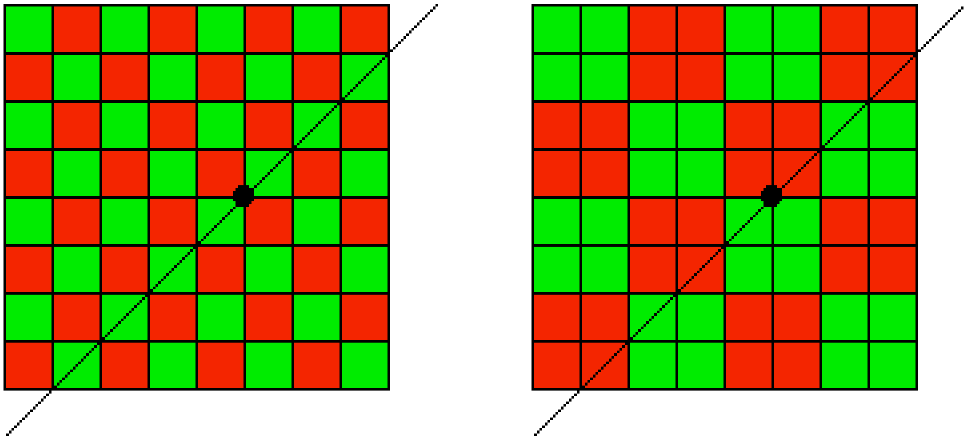

Fig. 9.4

These ‘infinite chessboards’ are for all practical purposes identical, but, for example, the indicated diagonal reflection and 90^0 rotation are consistent with color only in the case of the coloring on the left; still, even the coloring on the right has **enough** diagonal reflections and 90^0 rotations (etc) consistent with color for the corresponding tiling to be classified as a $\mathbf{p'_c4mm}$ (just as the one on the left, where the coloring has ‘preserved’ all the isometries).

It is possible though to have non-maplike colorings that yield no color inconsistencies. Consider for example the following two colorings:

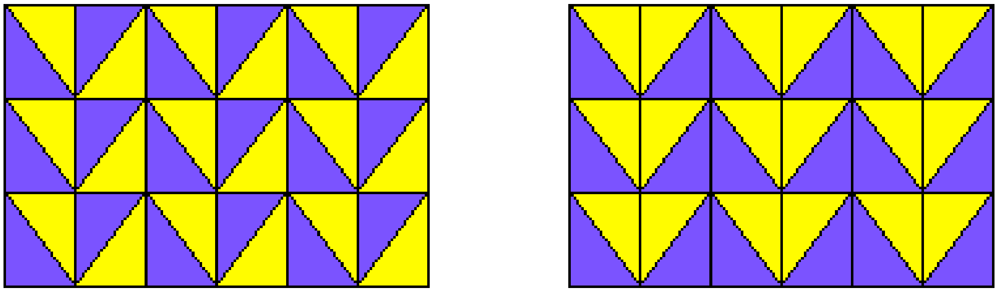

Fig. 9.5

The tiling on the left is a **pm′g**, while the one on the right is a **pmg′**. No isometry is inconsistent with color in either tiling, but the **pm′g** coloring on the left has ‘preserved’ the shape of the initial tiles, while the **pmg′** coloring on the right creates the impression of tiles in the shape of equilateral, rather than right, triangles.

The examples in figures 9.4 and 9.5 suggest that, in one way or another, part of the tiling’s original structure is lost when its coloring is not maplike. For this reason **all** colorings from here on, unless otherwise mentioned, are going to be **maplike**.

When, in addition to being maplike, a coloring is such that **all** the tiling’s isometries are **consistent** with color, the coloring is called **faithful** (to both the tiling’s tile shape(s) and isometries). For example, the two colorings on the left in figures 9.4 and 9.5 are faithful, while the ones on the right are not.

9.0.7 <u>Inconsistency and reduction of symmetry.</u> It is certainly possible for a maplike coloring to render some of the tiling’s isometries inconsistent with color. We will see several such colorings in the rest of the chapter, but here is an inconsistent 3-coloring of a **p4g** tiling, inspired by a stairhead ancient mosaic at the Bardo Museum in Tunis:

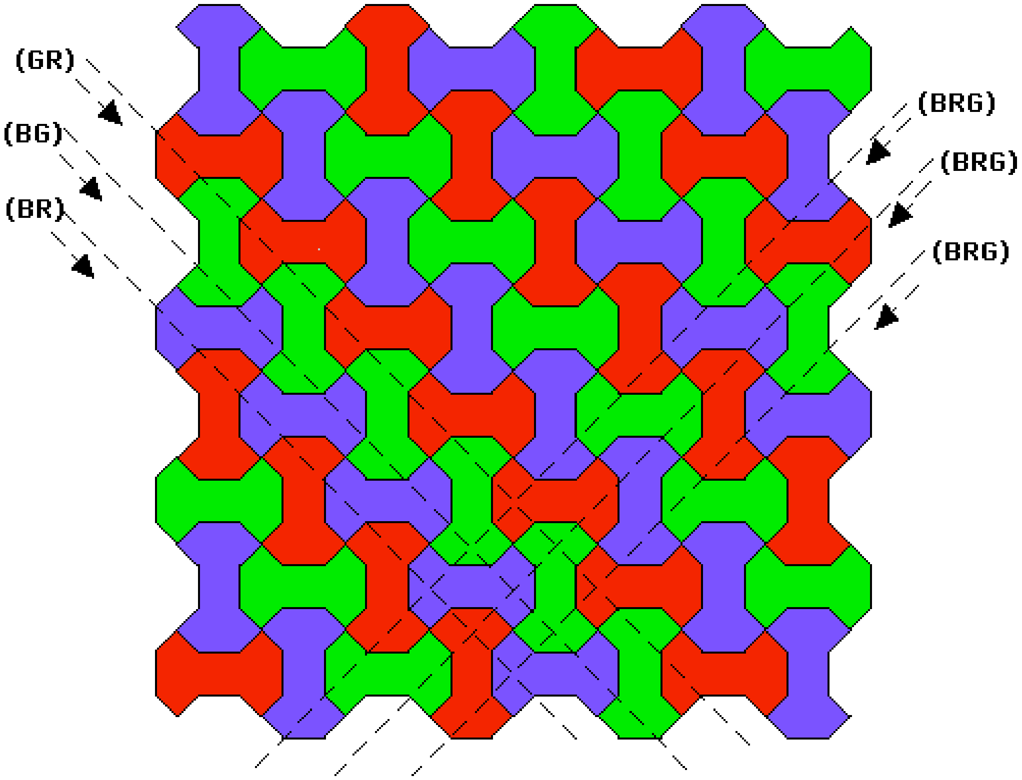

Fig. 9.6

All **90^0** rotations are now **inconsistent** with color, and same applies to all vertical-horizontal reflections and glide reflections; and yet the 90^0 centers do **still** host **consistent** with color **180^0** rotations (each of which swaps two colors and preserves the third), and the **p4g**'s diagonal glide reflections have also survived, even though NW-SE and NE-SW glide reflections behave differently when it comes to color effect (see figure 9.6): one could still say that the **p4g**'s underlying **vertical-horizontal cmm** structure is **gone**, while the underlying **diagonal pgg** structure has been **preserved**.

In other words, **unfaithful** colorings tend to **reduce** symmetry in the case of three or more colors, precisely as they did in the case of two colors (see examples in 6.4.2, 6.6.1, 6.9.5, 6.17.5). Of course in the case of three or more colors we will not 'measure' this reduction of symmetry by way of 'higher' and 'lower' symmetry types analogous to those introduced in chapter 6: symmetry types for multicolored patterns would be beyond the scope of this book.

In come cases the presence of inconsistencies in a coloring does

not lead to a reduction of symmetry (and a ‘lower’ symmetry type): look for example at the ‘enlarged chessboard’ of 9.0.6 or at the **p'_c4gm** patterns discussed in 6.12.4.

Back to the **p4g** tiling in figure 9.6, we must say that such symmetrical yet unfaithfully colored tilings are rare in the real world: once again, native artists had good symmetry instincts despite the lack of formal mathematical training! Indeed a two-color faithful coloring of this particular tiling is easy to find and was rather common in the Roman and Arabic eras; see for example figure 5.258 in ***Washburn & Crowe*** (from Fede Berti’s ***Mosaici Antichi in Italia***) for a two-color version of the Bardo mosaic from Ravenna.

9.0.8 <u>The art of M. C. Escher.</u> Much closer to our era, there was a Dutch artist who claimed no knowledge of mathematics and yet he stunned mathematicians, crystallographers, and just about everybody else with his faithfully colored tilings and their interlocking fish, birds, lizzards, butterflies, and other fauna: that’s **Maurits Cornelis Escher** (1898-1972), one of the most influential contemporary artists, celebrated for example in Douglas Hofstadter’s bestseller ***Godel, Escher, Bach: An Eternal Golden Braid*** (Basic Books, 1979 & 1999). For our purposes, a more useful work to which we often refer here is Doris Schattschneider’s definitive ***M. C. Escher: Visions of Symmetry*** (Freeman, 1990; Abrams, 2004): that’s the book you must read if you truly wish to understand Escher’s work! Also mathematically inclined, but of a broader scope, is ***M. C. Escher: Art and Science*** (North Holland, 1986), a collection of essays on Escher’s heritage edited by H. S. M. Coxeter, M. Emmer, R. Penrose, and M. L. Teuber; also, ***M. C. Escher’s Legacy: A Centennial Celebration*** (Springer, 1998), edited by D. Schattschneider and M. Emmer. And there are of course several other books (and web pages) that are more artistically oriented.

9.0.9 <u>A fivefold dream come true?</u> Look at the following design:

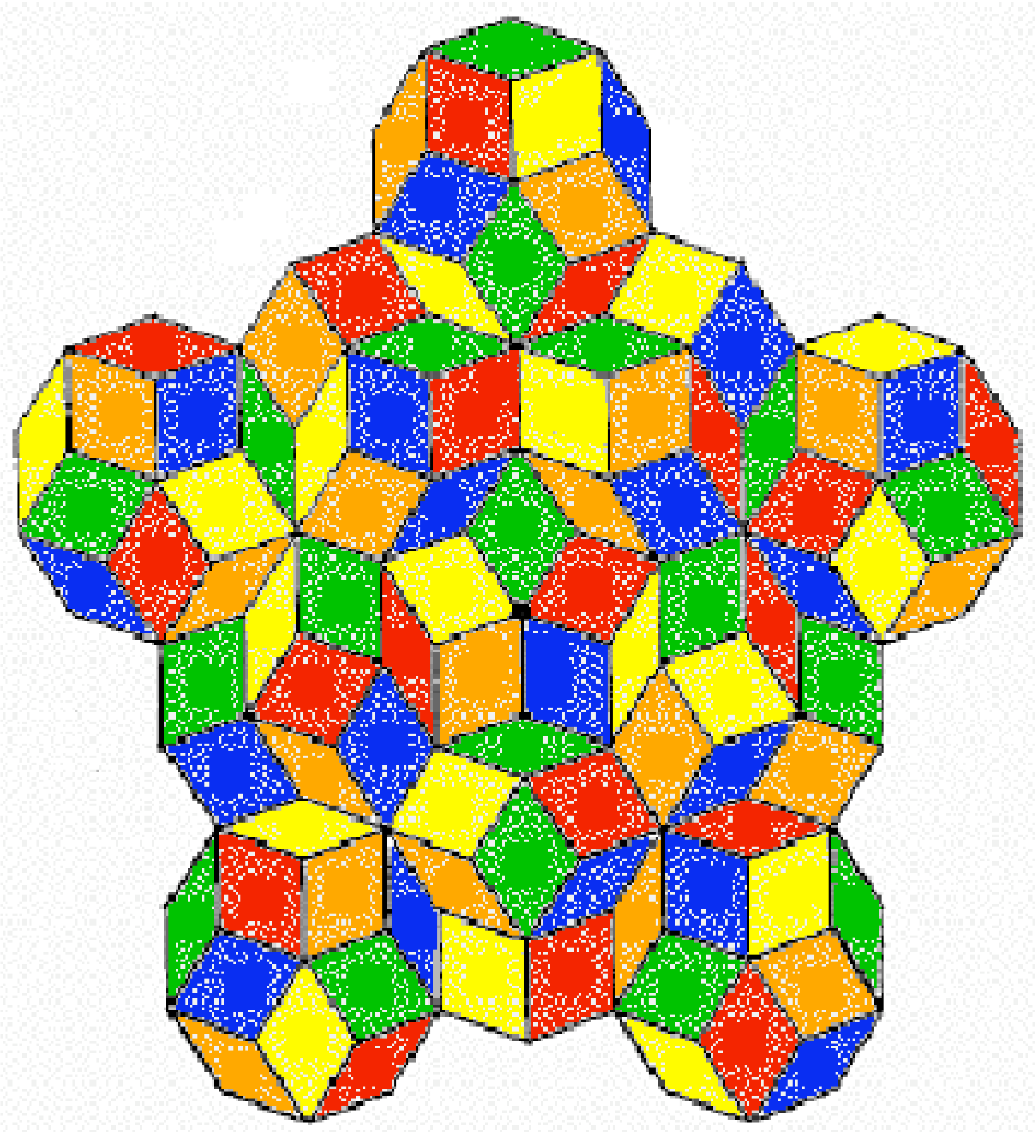

Fig. 9.7

What can we say about it based on what we have learned so far? Ignoring the coloring at first, we notice that it's a **finite** design of $\mathbf{D_5}$ type: indeed it has **fivefold rotation** about its center, and **reflection axes** in five directions, one of them running through its 'top point' and center. And, taking coloring into account, we also notice that it is faithfully colored in five colors as shown in figure 9.7.

So far so good: it's quite complicated, but it works after all! But here comes a surprise: it works **'for ever'**, being in fact an **infinite** design. That is, it turns out that we can keep adding rhombuses to it -- you must have noticed by now that our example consists of rhombuses of just **two kinds** -- for ever, preserving its **'central'** fivefold rotation, **five** reflection axes, and faithful coloring discussed above! **How** we can go on on such an endless journey towards infinity is a rather long story, but it's 'fact of life', or rather 'fact of nature': such **infinite D_5 designs**, constructed first (1974) by eminent British cosmologist Roger Penrose and called **Penrose tilings**, have also been confirmed experimentally (1982) by Israeli microscopist Daniel S. Schechtman and are known to natural scientists as **quasicrystals**. You may learn more about them (and their journeys to infinity) by reading Marjorie Senechal's monograph ***Quasicrystals and Geometry*** (Cambridge, 1996), although something like Malcolm W. Browne's New York Times article ***New Data Help Explain Crystals That Defy Nature*** (November 24, 1998) might well be a more realistic and appropriate starting point.

But, what is so surprising about them, and why are they called **quasi-**crystals, implicitly claimed to "defy nature"? Why such bad reputation? Well, you may recall that in section 4.0 we made a very indirect, but significant nonetheless, connection between crystals and wallpaper patterns by way of that **crystallographic restriction**: there are no wallpaper patterns (that is, infinite designs with translation in at least two non-opposite directions) having rotation by angles other than 60^0, 90^0, 120^0, or 180^0. In particular, no wallpaper pattern can possibly have fivefold rotation (by $\mathbf{72^0}$), a fact that many medieval artists striving for infinite fivefold symmetry would probably have been grateful to know in advance!

So, what's the catch here? Did Penrose fulfill the **impossible dream** of some distant ancestors? Has a mathematical loophole allowing for wallpaper patterns ('crystals') with fivefold rotation been found? Not really... The infinite D_5 design of figure 9.7, as well as its famous **kites and darts** cousin (also designed by Penrose) and other infinite fivefold designs are **not** wallpaper patterns: they are simply **too irregular to have translation**! Such 'irregular' wallpaper patterns and tilings are called **aperiodic** (for having no 'period', that is repetition and translation), and they are very much the subject of advanced current research: see for example Charles Radin's ***Miles of Tiles*** (American Mathematical Society, 1999). Unless otherwise stated, all tilings in this chapter will have translation and be **periodic**, belonging indeed to one of the seventeen types discussed in chapter 4.

We bid farewell to the aperiodic tiling of figure 9.7 by justifying its characterization as an infinite $\mathbf{D_5}$ design: in conformity with $\mathbf{D_n}$ designs as discussed in section 3.6, it has **only one** rotation center and **only one** reflection axis in each of its five directions of reflection. Indeed the existence of more than one rotation centers for the same rotation angle (in this case 72^0) would imply the existence of **translation** -- recall our observation in 6.10.1 or see discussion in 7.5.2 -- which the design in figure 9.7 does **not** have (as already stated above); and the existence of more than one reflection axes in any given direction would lead to **intersections** of reflection axes, therefore, as pointed out in 6.14.1 or 7.2.2, to **additional** rotation centers (that do **not** exist).

9.0.10 How many colors? For about the same reasons that **no** tiling with 120^0 rotation can be **faithfully** colored in **two** colors (6.13.1), there exist **no** faithful coloring of the **p4g** tiling of figure 9.6 in **three** colors or of the **aperiodic** tiling of figure 9.7 in **less than five** colors. But, as we should be able to see throughout chapter 9, all **periodic** (wallpaper) tilings discussed here may be **faithfully** colored in **four or less** colors; this does not imply that **all** possible periodic tilings are faithfully colorable in four or less colors, but it leads to a **conjecture** remininscent of the Four-Color Theorem:

?Every periodic tiling is faithfully colorable in four or less colors?

An illustration -- not proof! -- of this conjecture is provided in figure 9.8 below: a reasonably complicated **p4** pattern has been faithfully colored in four colors!

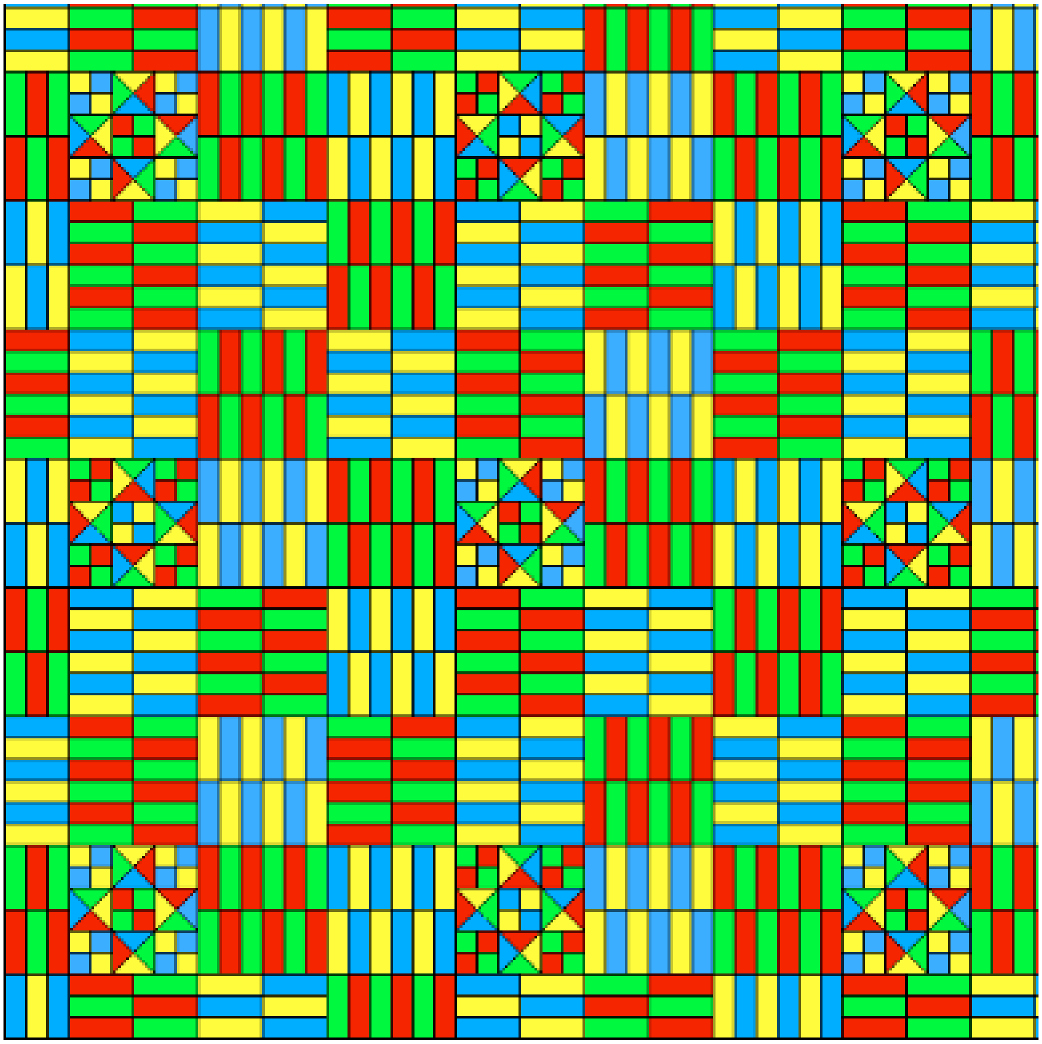

Fig. 9.8

[Sections 9.1 through 9.17 will hopefully be added at a future date]

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