Shredded Wallpaper

A honors thesis submitted under the title

“The Tiling of Wallpaper Patterns by Border Patterns”

by

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Introduction

We live in a world of patterns. This is demonstrated in books such as Dorothy Washburn and Donald Crowe's *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis* and Doris Schattschneider's *M. C. Escher: Visions of Symmetry*. I will be investigating whether a given wallpaper pattern can be tiled using a given type of border pattern, as touched on in SUNY Oswego's Symmetries (MAT 103) and George Baloglou's *Isometrica*.

My topic represents a small section of Symmetries. Symmetries is the mathematical study and classification of patterns through analyzing the different distance preserving transformations (also known as isometries) which can be applied to the pattern without visibly altering it. I will provide an overview of those principles in Symmetries which are necessary in order to understand my study. For additional information on these topics, the reader is referred to the books mentioned above.

Mathematicians who work in Symmetries classify patterns of two different types, border patterns and wallpaper patterns. A border pattern has "one and a half" dimensions, with a constant width and an infinite length. There are seven distinct types of border patterns. A wallpaper pattern is a two dimensional pattern which spans the plane. There are seventeen types of wallpaper patterns.

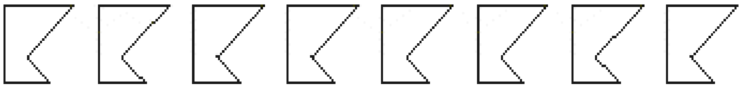
Both types of patterns lie on a plane and have translation. I will use the term "strip" to refer to a section of a wallpaper pattern which has finite width and infinite length, but which does not necessarily have a translation vector, and thus may not be a border pattern of any type (Appendix 1). A border pattern has only one direction of translation, parallel to the direction of the pattern itself, while a wallpaper pattern must have translation in two distinct directions and will, as a result, have translation in infinitely many directions. For now, assume that the existence of any translation vector on a wallpaper pattern implies the existence of some translation vector perpendicular to itself (Appendix 2). The qualities that differentiate between types of border patterns or wallpaper patterns are the isometries of the plane: reflection, rotation, glide reflection and translation. As isometries, all these transformations maintain size and distance, but may change placement and orientation. The classification of a given pattern is derived by considering the existence of different isometries and the interaction between them.

Border Patterns

I will now define the various types of border patterns, by breaking apart the titles into their descriptive elements. All border patterns have the letter p in the first position, which stands for "pattern". The p shows that the pattern has a translation vector and is not simply a strip. The second position describes vertical reflection, the third horizontal reflection or glide reflection, and the fourth describes 180° rotation.

The simplest type of border pattern is the p111, which contains no isometries other than the translation defined above. A 1 in any position shows that there is no isometry of that particular type.

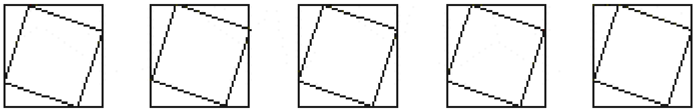
An example of a p111



This pattern has no isometries except a translation whose length is from the start of one shape to the start of the next.

A pattern with a 2 in the fourth position of the name, in effect, pma2, p112 and pmm2, will contain 180° rotation, also referred to as half turn. This transformation gives each of these patterns infinitely many rotation points around which one can rotate the entire pattern by 180°, without altering the pattern. The centers of the rotation must lie on the center line of the border pattern. The simplest border pattern of this type is the p112 that has only 180° rotation.

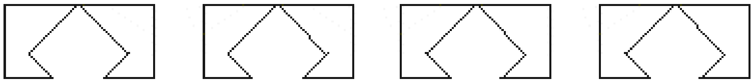
An example of a p112



The length of the translation in this pattern is the distance between centers of two adjacent squares, and rotation points exist in the centers, as well as directly between two squares, on the center line.

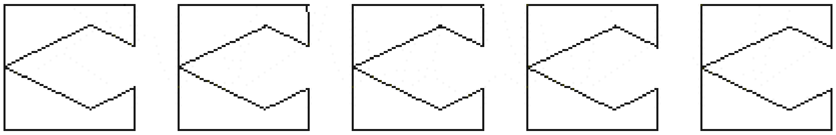
There are two types of reflection that can exist in a border pattern: vertical and horizontal reflection. The line representing horizontal reflection must coincide exactly with the center line on the pattern. Vertical reflections are perpendicular to the direction of the border pattern and must be evenly spaced along the center line. These transformations are represented in the patterns' names with the letter m, placed in the second space for vertical reflection, and the third space for horizontal reflection. For example the pm11 has only vertical reflection and the p1m1 has only horizontal reflection. The pmm2 pattern is the only one containing both reflections, as well as 180^{o} rotation points on each intersection of its horizontal with vertical reflection axes.

An example of a pm11



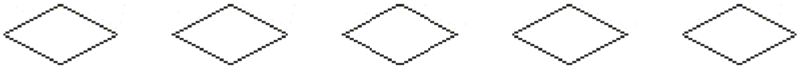
This pattern has a vertical reflection axis in the center of each "gate" and also directly between two adjacent gates.

An example of p1m1



This pattern has a horizontal reflection axis running directly through the center of the gates.

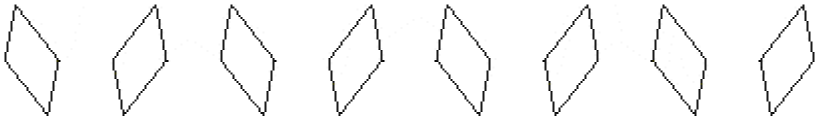
An example of a pmm2



Here the vertical reflection axes lie in between diamonds as well as along the center of each diamond. The horizontal reflection axis lies on the center line, running through each diamond, and the rotation points lie on the intersections of reflection axes, in the center of each diamond as well as directly between two diamonds.

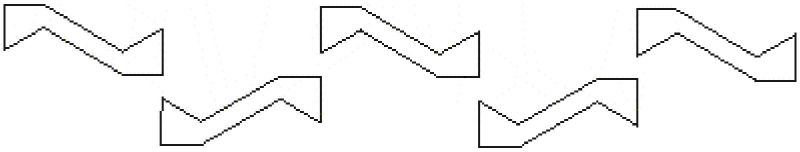
The final type of isometry in border patterns is the glide reflection, by which the pattern flips over a reflection axis and then translates by a parallel vector. This axis also will appear only on the center line of a border pattern. It is noted by the letter a appearing in the third spot in the title, as in pma2 and p1a1. The p1a1 has glide reflection only, while the pma2 has glide reflection combined with vertical reflection and half turn rotation points lying half way in between two vertical reflection axes. Though clearly a pattern that has horizontal reflection will always also have glide reflection, horizontal reflection is the more notable case. When naming a border pattern horizontal reflection is considered before glide reflection.

An example of a pma2



The vertical reflection lies in between pairs of adjacent parallelograms and the rotation points lie on the center of each parallelogram. The glide reflection, of course, runs through the center line.

An example of a p1a1



Again, the glide reflection here runs through the center line.

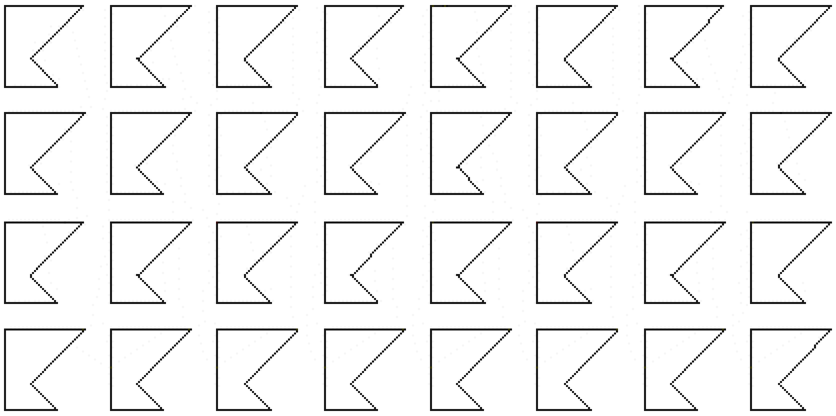
By understanding the way in which these patterns are named one can easily infer, from the name of a pattern type, which isometries it contains. There are seven types of border patterns: p111, p112, p1m1, pm11, pmm2, pma2, and p1a1.

The system of assigning names to the seventeen wallpaper patterns is not as logical, and so I will define the types of wallpaper patterns below without reference to notation.

Wallpaper Patterns

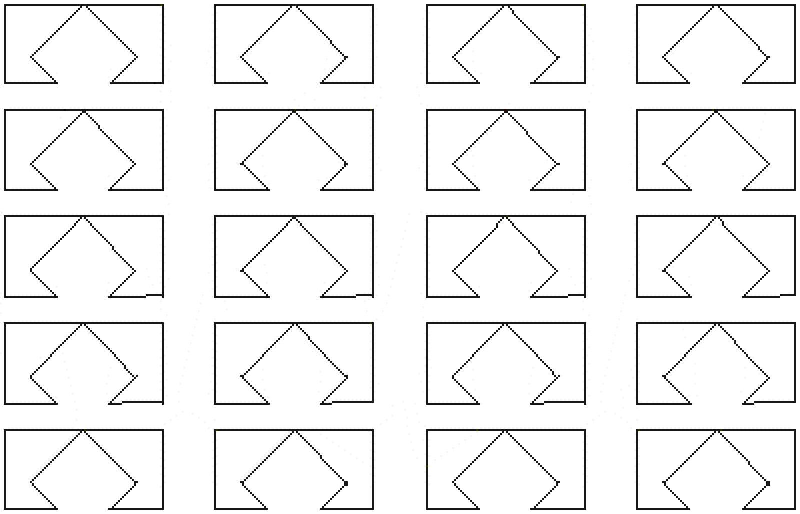
Wallpaper patterns are classified in groups based on the degree of smallest rotation in the pattern. The patterns with none, only 360° rotation, are named p1, pm, pg, and cm. The p1 has only translation, the pm has two types of parallel reflection, the pg has two types of parallel glide reflection and the cm has alternating and parallel reflection and glide reflection axes.

An example of a p1



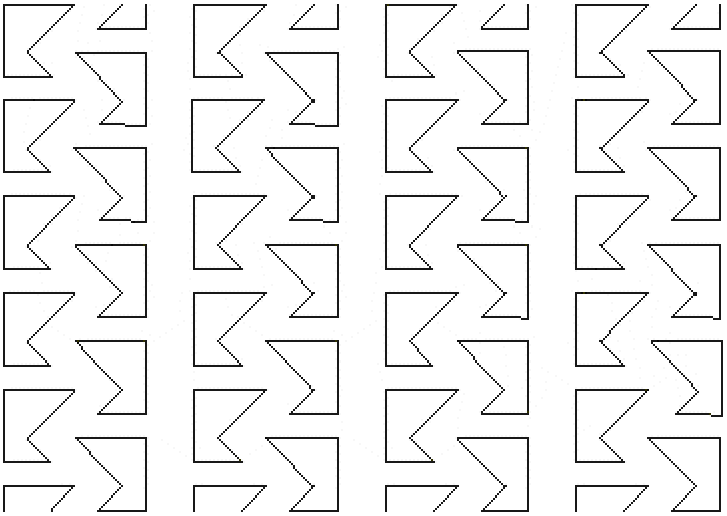
In addition to the obvious vertical and horizontal translation, there is also slanted translation in infinitely many directions. A translation vector is any vector which will take a point on one shape to an equivalent point on a congruent shape.

An example of a pm



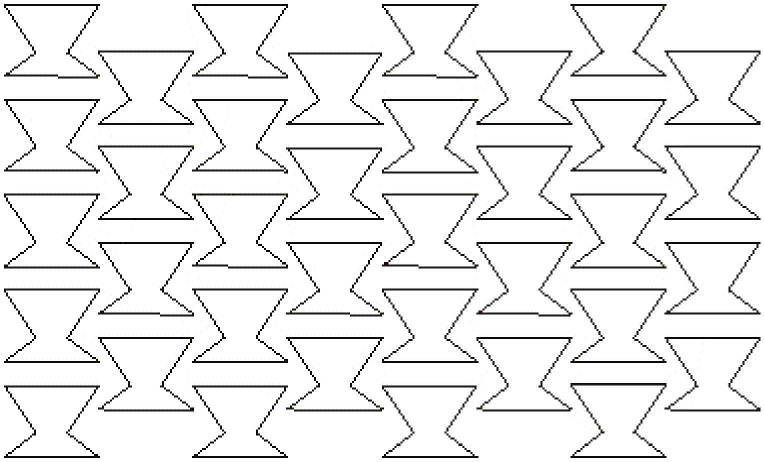
This pm has two kinds of vertical reflection axes, one running through the centers of the gates and one kind directly in between adjacent gates.

An example of a pg



Similar to the above pattern, this has two kinds of glide reflection axes. One is between columns and one is through the centers of the broken gates.

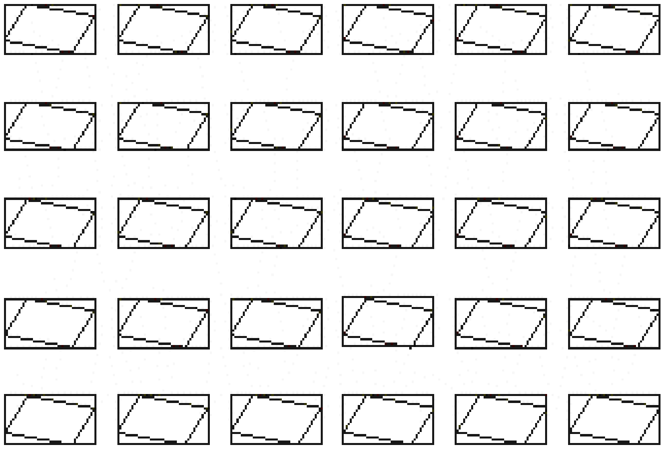
An example of a cm



The reflection axes here run though the centers of the goblets and glide reflection axes run between columns of goblets.

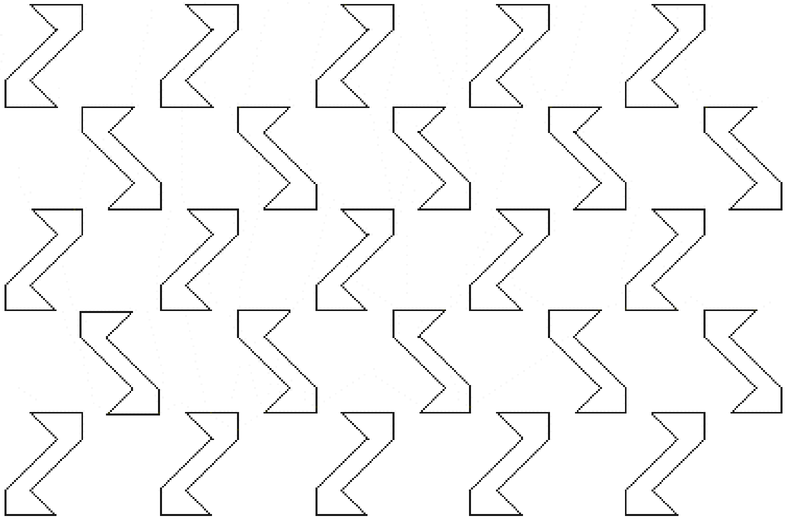
Patterns with smallest rotations of 180° include the p2, which has only rotation, and the pgg, which has glide reflection in two perpendicular directions with rotation points half way between these axes. The pmm pattern has reflection in two perpendicular directions, and all rotation points lie on the intersection of two axes. The cmm has both perpendicular glide reflection axes and perpendicular reflection axes with rotation points lying at the intersection of two like axes. Finally, the pmg has reflection axes in one direction and glide reflection axes perpendicular to the reflection. All of the rotation points on a pmg lie on glide reflection axes, half way between two reflection axes.

An example of a p2



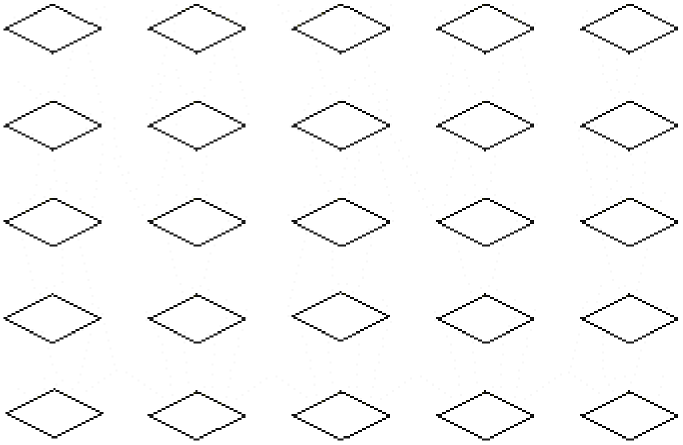
The four kinds of rotation points lie on the centers of the boxes, and directly in between two boxes that are adjacent either vertically, horizontally or diagonally.

An example of a pgg



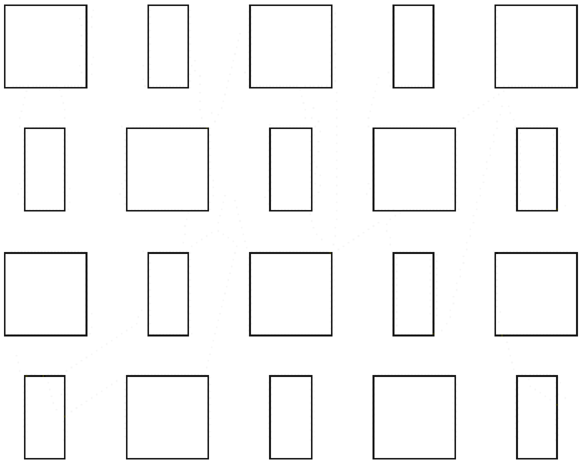
There are two kinds of perpendicular glide reflections, each running in between vertical columns or horizontal rows of zigzag shapes. The rotation points lie on the centers of these shapes, and directly in between them, either vertically or horizontally.

An example of a pmm



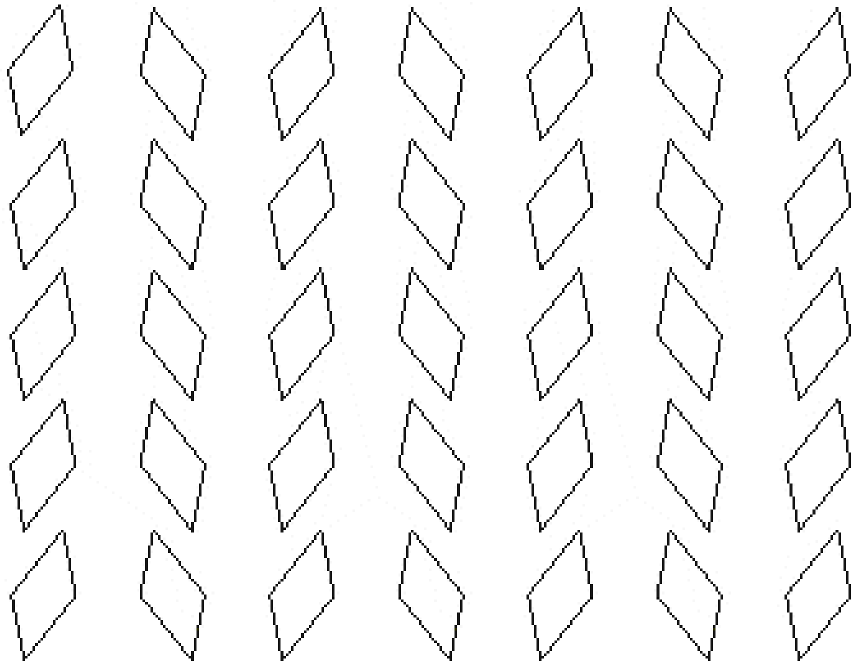
The four kinds of reflections here quite clearly run through and between rows and columns of diamonds. Rotation points lie on intersections of two reflection axes.

An example of a cmm



Reflection axes run through rows and columns of shapes; glide reflection axes run between them and rotation points lie on the intersection of either two reflection axes or two glide reflection axes.

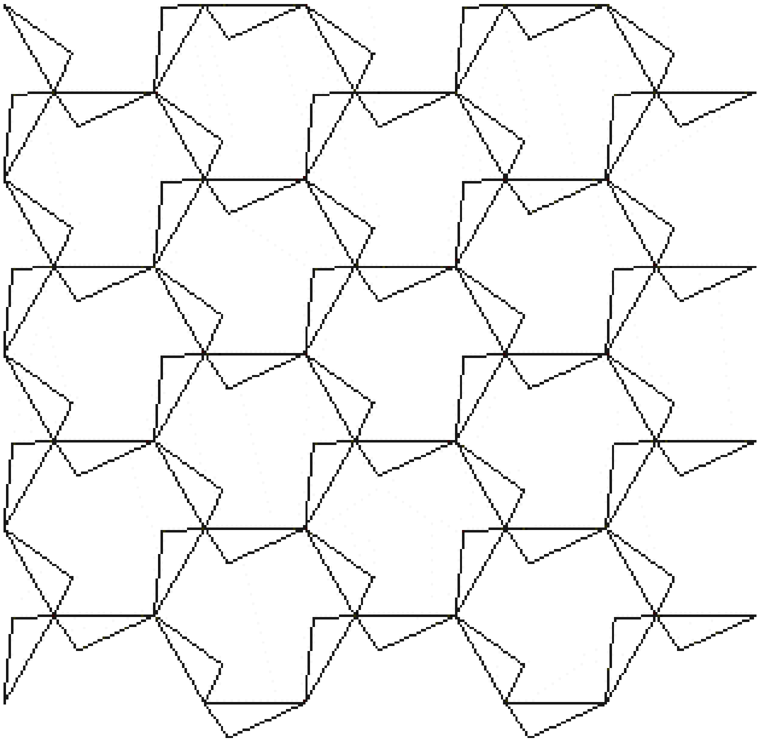
An example of a pmg



Though this pattern appears to have two kinds of vertical reflection axes, the two kinds are actually images of each other, created by the half turn. Each vertical reflection axis runs half way between two columns of parallelograms. Horizontal glide reflection runs through tops and bottoms or centers of parallelograms. Rotation points lie on the glide reflection axes, directly between two reflection axes. In this example, they are in the centers of and directly in between parallelograms.

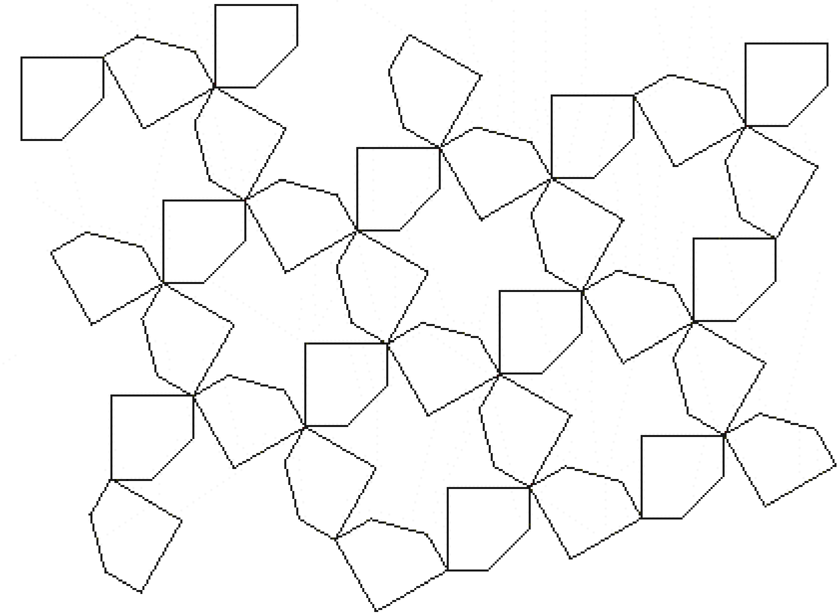
The basic threefold, or 120° rotation wallpaper pattern is p3, with only rotation. The p3m1 has three directions of reflection, with 120° rotation points lying on the intersection of three axes, and three directions of glide reflection in between the reflection axes. The p31m has isometries identical to the p3m1, except that the p31m also includes rotation points that do not lie on any reflection axes.

An example of a p3



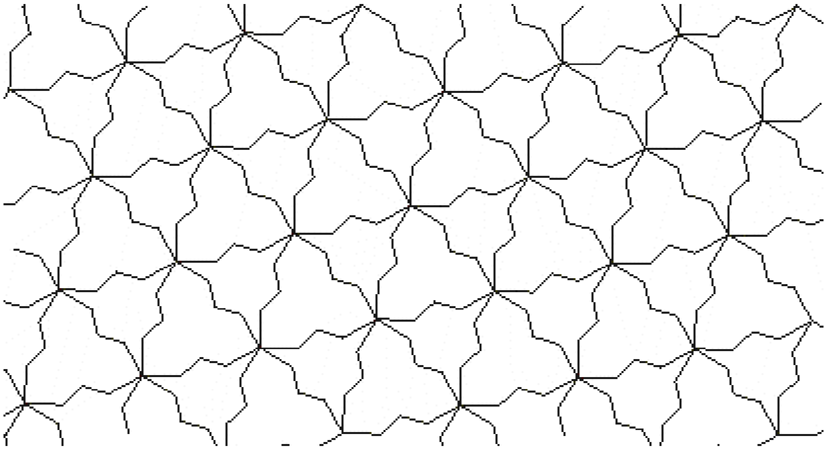
The three kinds of rotation points in this pattern are easy to see in the center of each hexagon, and the centers of the pinwheels.

An example of a p31m



In this pattern, the rotation points which do not lie on reflection axes sit on the centers of the pinwheels, while the reflection axes which cut the pentagons in half intersect in the center of each "flower" to form the other type of rotation points.

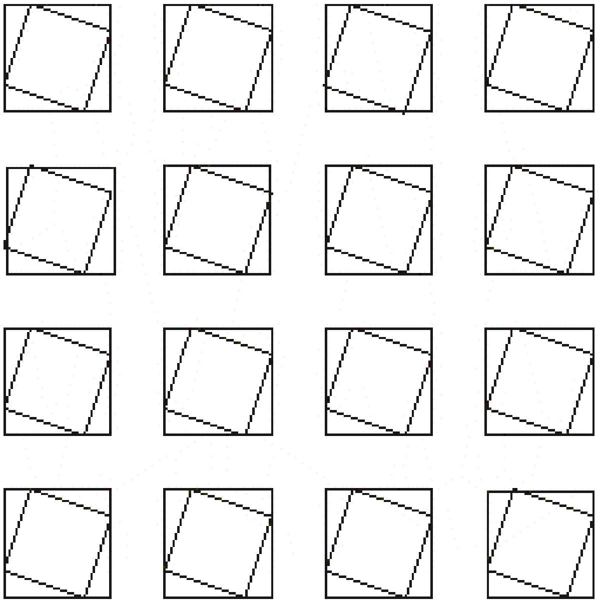
An example of a p3m1



The three directions of equivalent reflection axis here are easy to see, and all rotation points lie on the intersection of three axes.

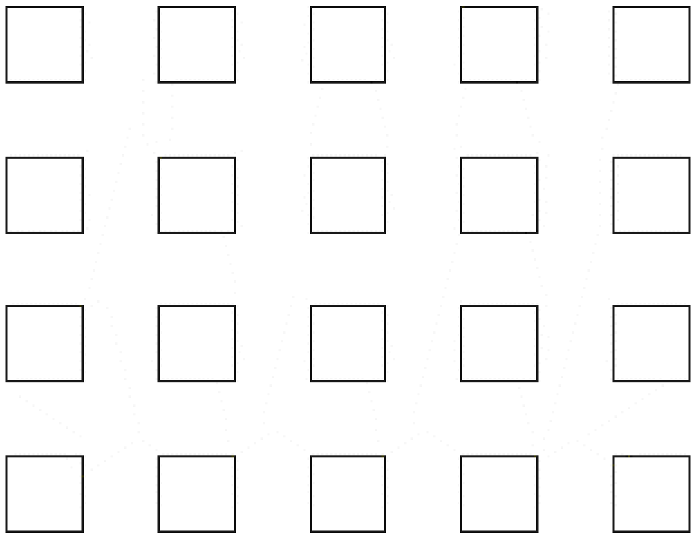
Patterns with smallest rotation of 90° or fourfold rotation are p4, p4m and p4g. The p4 is the pattern with only 90° and 180° rotations. A p4m will have four directions of reflection and two directions of glide reflection, with 90° rotation points falling on the intersection of four reflection axes, and with 180° rotation points falling on the intersection of two reflection and two glide reflection axes. The p4g pattern has four directions of glide reflection and two directions of reflection. Its 180° rotation points lie on the intersection of two perpendicular reflection axes and 90° rotation points lie on the intersection of two perpendicular glide reflection axes parallel to reflection axes.

An example of a p4



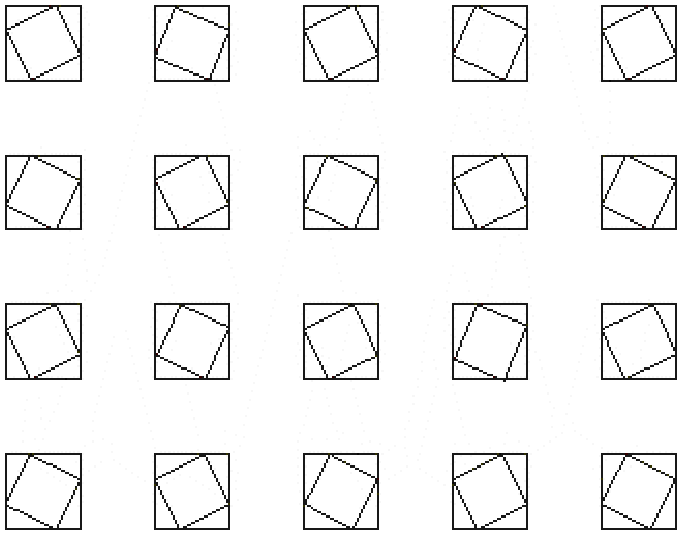
Rotation points in this pattern lie in the same locations as the rotations in the p2. The 90° rotation points are the points in the centers of the squares and at the points where four corners come together. All other rotation points are 180°; note that all these rotation points are equivalent, since they can be brought to one another by either a translation or a 90° rotation.

An example of a p4m



Reflection axes run through the squares horizontally, vertically, and diagonally, as well as directly in between the rows and columns. Glide reflection axes pass between the boxes diagonally, connecting sets of vertices. Rotation points are in the same place and of the same type as in the previous p4 pattern.

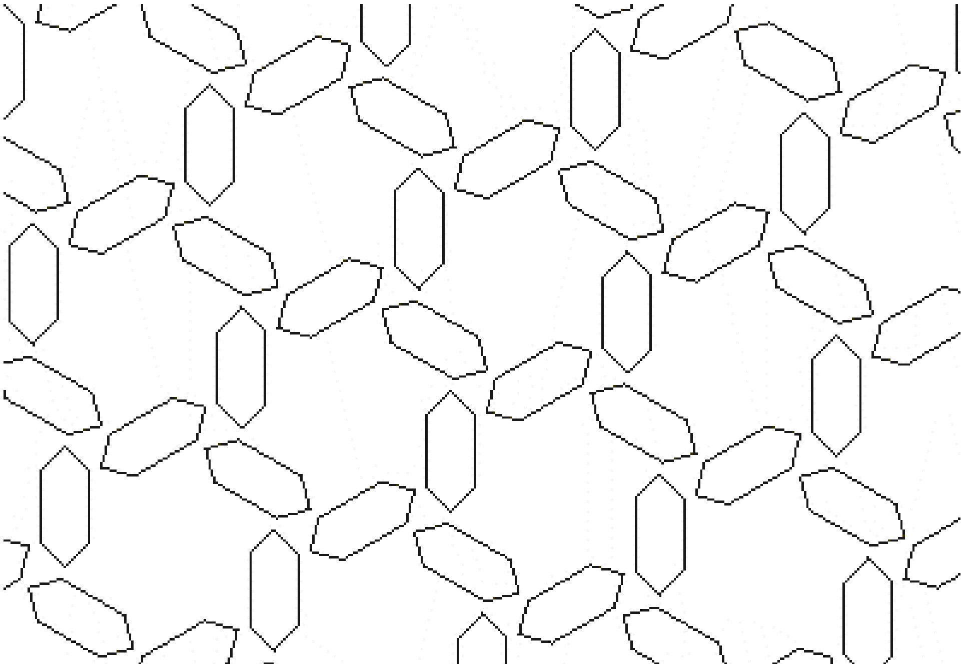
An example of a p4g



Here the 90° rotation points in the centers of the squares are identical to the previous two patterns, but the points that fall at the corners of four squares are instead 180° points. All 180° points from the previous patterns are gone. All glide reflection axes have remained from the p4m, as have the reflection axes between rows and columns. Reflection axes that divided squares in half have become glide reflection axes in the p4g.

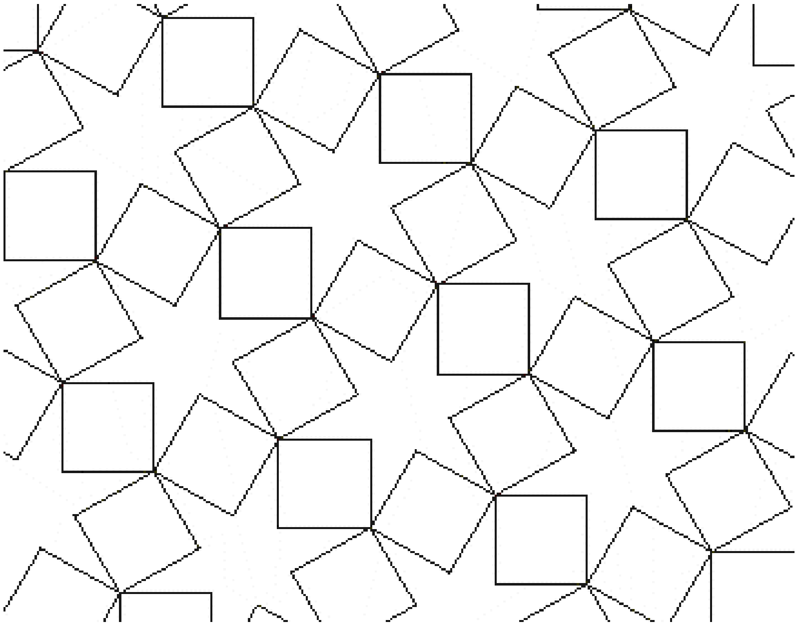
Finally, there are two types of pattern with 60°, or sixfold rotation, which is the smallest possible rotation on a wallpaper pattern. The p6 contains only rotation points of 60°, 120°, and 180°, and the p6m has all these rotations as well as six directions of reflection and glide reflection. In the p6m, the 60° rotation points lie on the intersection of six reflection axes, the 120° rotation points lie on intersections of three reflection axes, and the 180° rotation points lie on the intersection of four glide reflection axes and two reflection axes.

An example of a p6



Here, sixfold centers lie in the center of six hexagons, threefold centers lie in the center of three hexagons, and half turn centers are in the center of each hexagon.

An example of a p6m



Sixfold centers lie on the center of each star, threefold centers lie on the intersection of three squares and half turn centers lie in the center of each square. Reflection axes cut every square in both diagonal directions, and glide reflection axes run in between pairs of reflection axes.

The transformations that exist only in wallpaper patterns are 60°, 90°, and 120° rotations, or sixfold, fourfold and threefold rotations. Of these three, only 60° and 90° rotations will be important when considering the classification of strips contained within a wallpaper pattern. Either of these two rotations will also imply the existence of 180° rotation (through triple or double application, respectively) in the same location, while 120° rotation will not.

The Problem

I hypothesize that for each wallpaper pattern type there exists a concrete set of border pattern types that may tile the pattern. I will address each possible combination, explaining through examples and logical arguments why a certain border pattern may or may not tile a wallpaper pattern of some specific type.

In all examples of border patterns contained within wallpaper patterns, I will outline the border pattern in thick dashes and I will show the center, dotted line since in most cases the placement of the center line is the most important aspect of the pattern.

The following chart identifies the border patterns that will tile each type of wallpaper pattern. I will defend the results displayed in the chart with examples for positive results and arguments, usually in the form of small lemmas, for negative results. In conclusion, I will state and defend a set of qualifications required in order for a wallpaper pattern to contain some type of border pattern.

The possible tiling border patterns for any wallpaper pattern type.

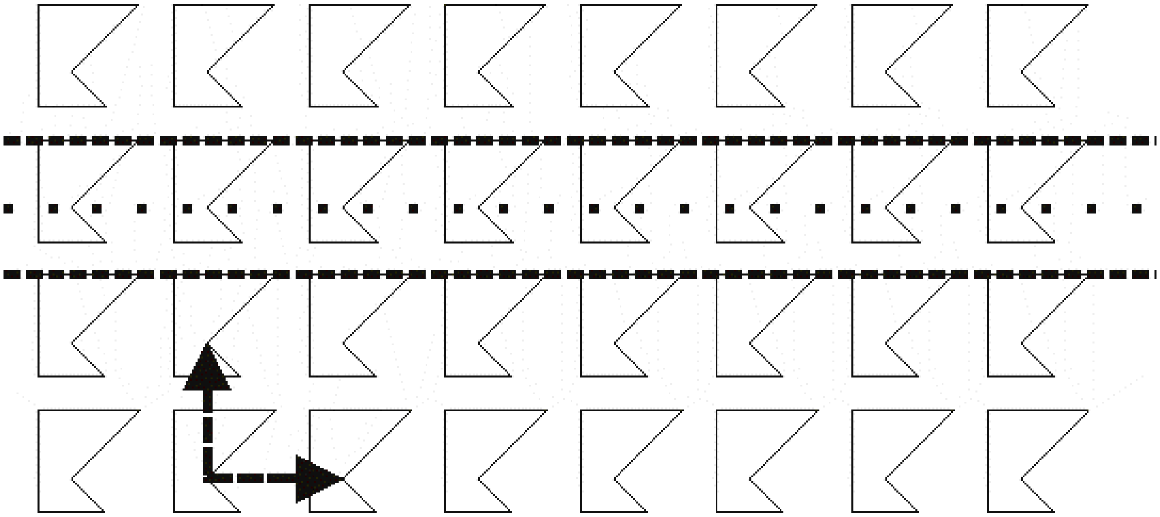
	p111	p112	pm11	p1m1	pmm2	pma2	p1a1
p1	yes*	no	no	no	no	no	no
pm	yes	no	yes	yes	no	no	no
pg	yes	no	no	no	no	no	yes
cm	yes	no	yes	yes	no	no	yes
p2	yes*	yes*	no	no	no	no	no
pgg	yes	yes	no	no	no	no	yes
pmm	yes*	yes*	yes	no	yes	no	no
cmm	yes*	yes*	yes	no	yes	yes	no
pmg	yes	yes	yes	yes	no	yes	no
p3	yes	no	no	no	no	no	no
p31m	yes	no	yes	yes	no	no	yes
p3m1	yes	no	yes	yes	no	no	yes
p4	yes	yes	no	no	no	no	no
p4m	yes	yes	yes	no	yes	yes	no
p4g	yes	yes	yes	no	yes	yes	yes
p6	yes	yes	no	no	no	no	no
p6m	yes	yes	yes	no	yes	yes	no

*These results are dependent on our assumption of perpendicular translation, when such vectors are not created by combinations of reflection or glide reflection axes, or merely rotations; see also Appendices 2 and 3.

The p1 pattern

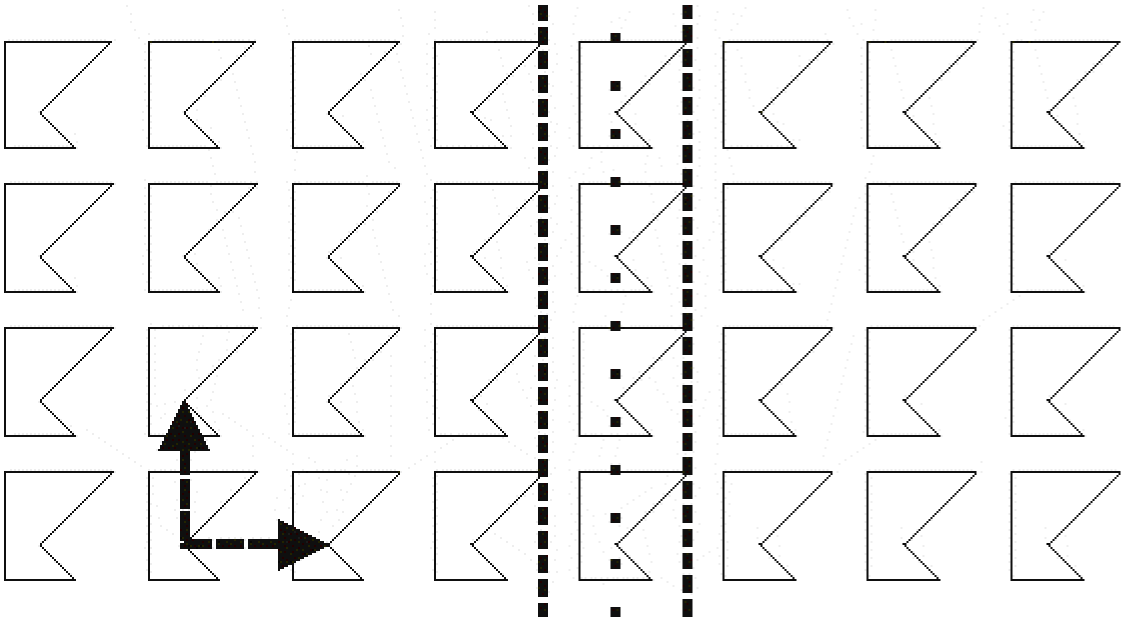
The existence of a p111 in the p1 can be shown in the following example. As for the other six border pattern types, we can show that they are absent in the p1 pattern using a set of lemmas that prove that it is impossible for a border pattern to contain any isometry that is not present in the parent wallpaper pattern. Thus the remaining types will not appear in a p1. First of all, in the following figures, there are a few obvious p111 patterns contained within the p1.

Figure 1a: An example of a p111 within the p1



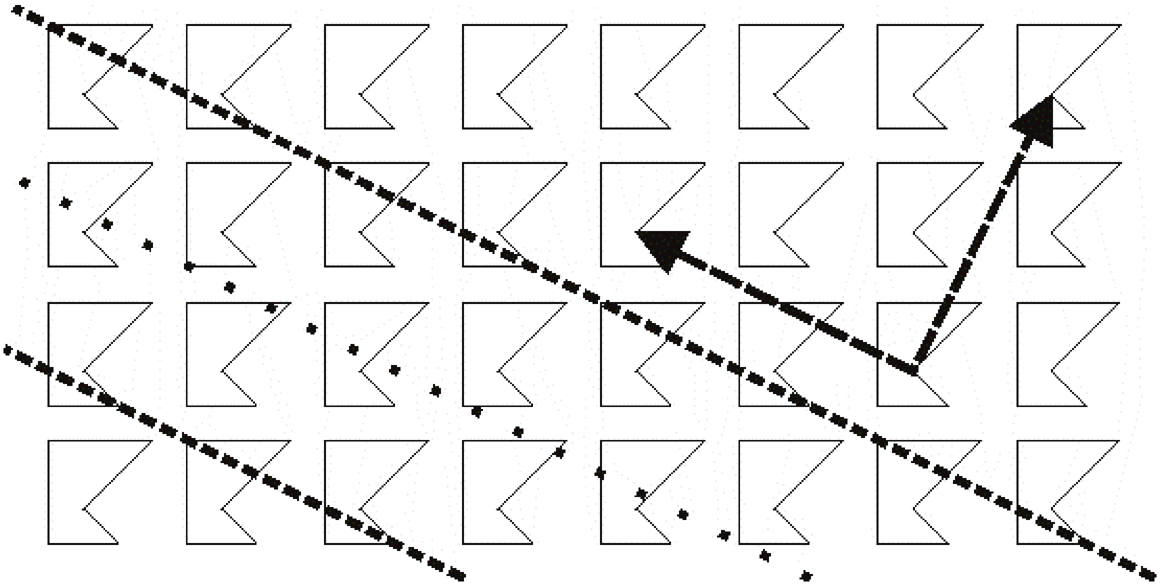
The p111 pattern found here is quite obvious, because it utilizes the smallest pair of translation vectors and also partially due to the fact that the p1 presented is so simple. The vertical translation vector shown here will describe not only the length, but also the way in which this p111, and any other border pattern in this direction, will tile the wallpaper pattern. The same pair of vectors of course could be used to create another, equally obvious, p111 pattern, as shown in figure 1b.

Figure 1b: A vertical p111 within the same p1



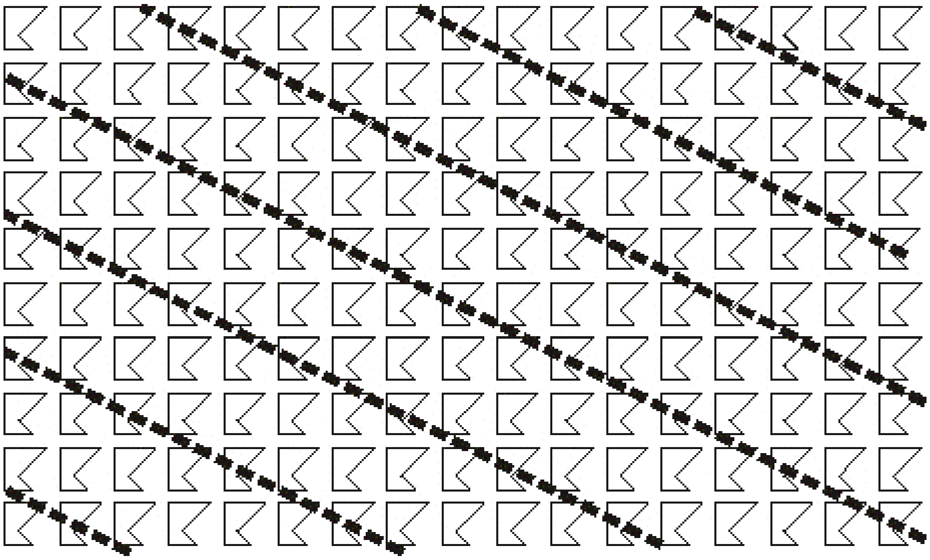
The vertical example is created using the same pair of translation vectors. Whereas in this example the horizontal vector describes the width and the way in which the border pattern may tile the p1, the horizontal translation vector in figure 1a functions as the translation within the border pattern. Here it is the vertical translation that becomes the translation within the wallpaper pattern.

Figure 1c: A more complex p111 within the same p1



So far I have shown two examples of p111 patterns within the p1. The border patterns in figures 1a and 1b are the more obvious border patterns within this wallpaper pattern. In those figures, it is apparent that the border pattern could be stacked so as to create the wallpaper pattern, since the shortest translation vector was used. The p111 in figure 1c could be stacked in the same way to recreate this p1, but it is not as easy to visualize.

Figure 1d: The tiling of the p1 by the border pattern in figure 1c

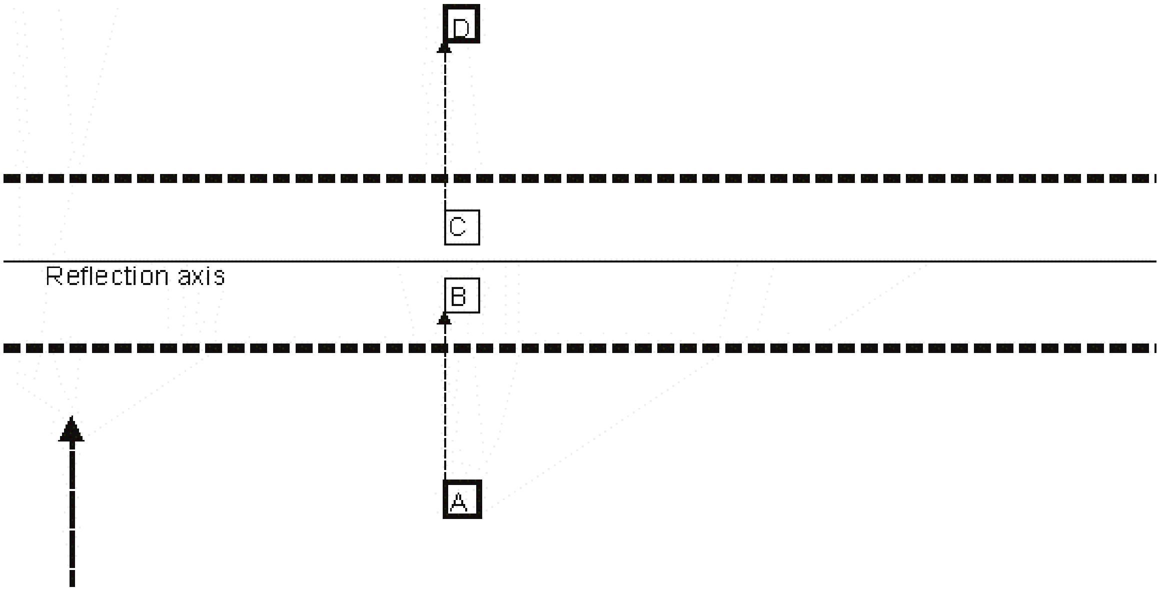


Since this border pattern does not have a clear structure in itself, it would be more difficult to identify within the wallpaper pattern. We will see many examples of patterns which are as easy to understand, as in figures 1a and 1b, and a few which can only be shown as relatively thick, complex border patterns, as in figure 1c.

Of course the differences between the three different examples of p111 patterns are superficial. All three patterns exist because of the existence of perpendicular translation vectors. Many wallpaper patterns will include border patterns that exist because of translation vectors that are created by interactions between reflection or glide reflection axes. This will be discussed further in the appendices.

I will now demonstrate the impossibility that any border patterns other than p111 may tile a p1, by creating a set of lemmas that will also be useful in the consideration of other wallpaper patterns. To begin demonstrating the necessity of isometries extending from a border pattern into the parent wallpaper pattern, I will present the case of horizontal reflection.

Figure 2: Extending horizontal reflection from border pattern to tiled wallpaper pattern



Lemma 1: If a border pattern tiles a wallpaper pattern and the border pattern has horizontal reflection then its horizontal reflection extends to the entire wallpaper pattern.

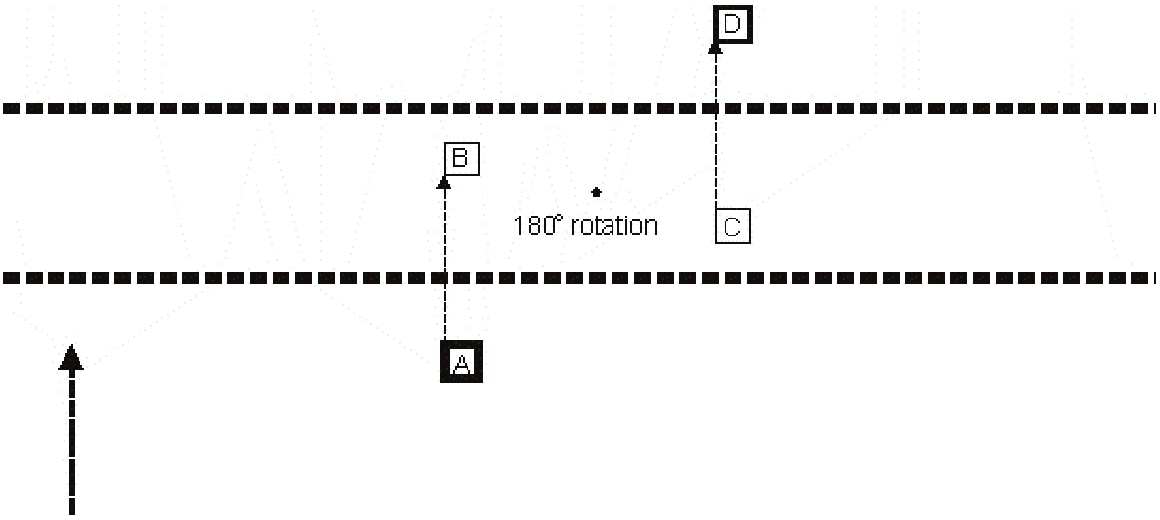
Proof:

As pictured in figure 2, the horizontal reflection axis is assumed to effect objects only within the border pattern. The vertical translation vector, which by assumption is equal to the width of the border pattern, affects the entire wallpaper pattern. For all points A on the wallpaper pattern, this vector, or an integer multiple of it, will take A into the border pattern. In figure 2, the image of A after translation is B. Since B is within the border pattern we can reflect the point B to C. When the point after reflection, C, is translated by the same vector, as shown, we are left with the image D. Note that the point D is identical to the image which would result from simply reflecting A over the reflection axis. We can see from this that a reflection axis which lies on the center line of a border pattern within a wallpaper pattern will apply to the parent wallpaper pattern as well.

This lemma allows us to say that in a wallpaper pattern without any reflection, there will be no border patterns which contain horizontal reflection, and further that in any wallpaper pattern, no tiling border pattern can have a horizontal reflection axis which is not also a reflection axis of the wallpaper pattern.

We can thus conclude that the p1 wallpaper pattern contains no pmm2 or p1m1 border patterns. I will continue with the other isometries of a border pattern in order to show that no border patterns other than the p111 will exist within the p1.

Figure 3: Extending half turn from border pattern to tiled wallpaper pattern



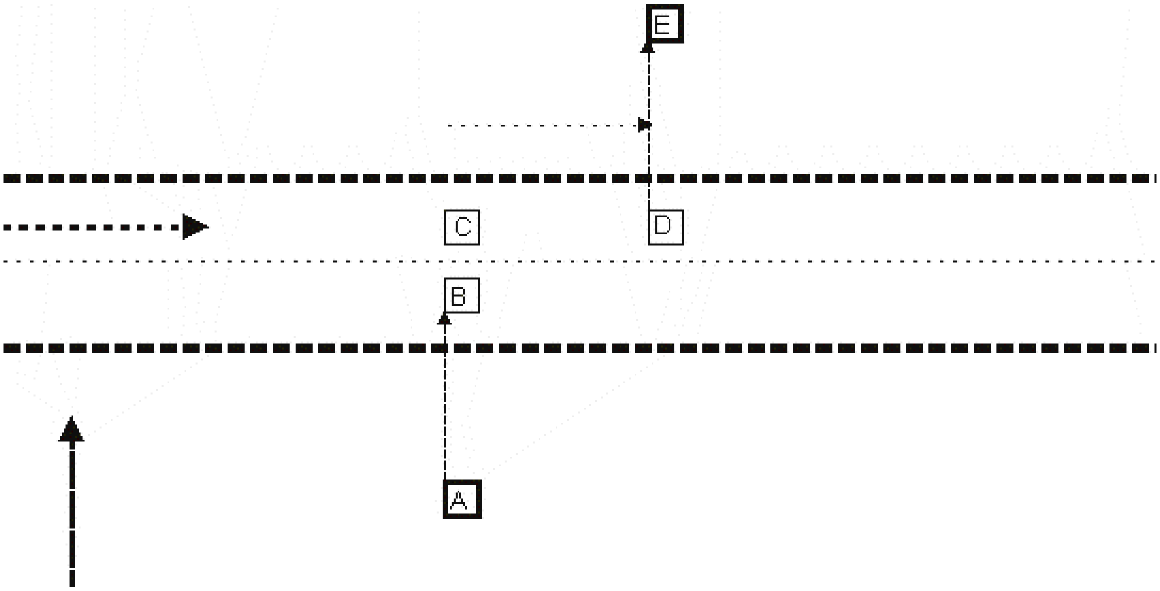
Lemma 2: If a border pattern tiles a wallpaper pattern and the border pattern has half turn then its half turn extends to the entire wallpaper pattern.

Proof:

In figure 3 the point A is translated by the vertical translation, or some integer multiple of it, to the point B within the border pattern. B is rotated to the point C, which is then translated to point D. Note that D would also be the image of A when rotated 180° about the half turn center. Thus rotation within a border pattern generates a rotation for the parent wallpaper pattern.

We can now conclude that the p1 pattern contains no p112 border patterns. It should also be clear that any border pattern will have half turn points only if the parent wallpaper pattern has 60°, 90° or 180° rotation points at these points. (We know that 60° and 90° rotation points will act as half turns since they are divisors of 180°.)

Figure 4: Extending glide reflection from border pattern to tiled wallpaper pattern



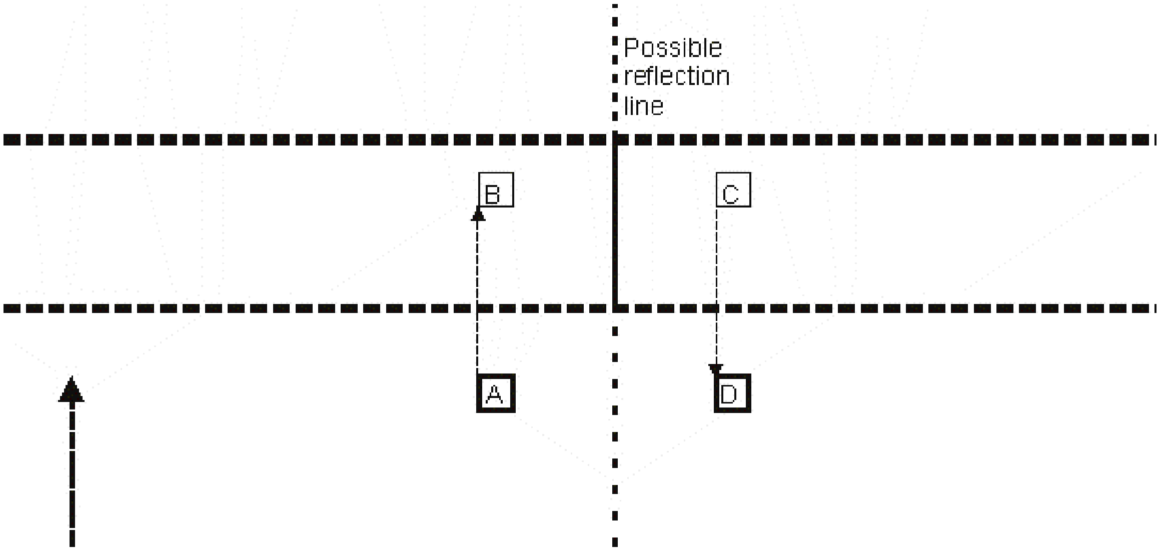
Lemma 3: If a border pattern tiles a wallpaper pattern and the border pattern has glide reflection then its glide reflection extends to the entire wallpaper pattern.

Proof:

In figure 4 the point A is translated, as in figures 2 and 3, into the border pattern, to the point B. The glide reflection reflects point B to C and glides point C to D, which is then translated by the original translation vector to the point E. Since E is equivalent to the image of A glide reflected across the center axis, we can see that in order for a glide reflection to work on a border pattern, it must also work on the wallpaper pattern which contains it.

Since a p1 pattern has no glide reflection, it will clearly contain no p1a1 or pma2 border patterns.

Figure 5: Extending vertical reflection from border pattern to tiled wallpaper pattern



Lemma 4: If a border pattern tiles a wallpaper pattern and the border pattern has vertical reflection then its vertical reflection extends to the entire wallpaper pattern.

Proof:

Figure 5 shows how a vertical reflection, like all other isometries within a border pattern, will function on the entire wallpaper pattern if it functions on a border pattern contained within it. Here the point A is translated, as in all other figures, to some point B within the border pattern. Then the point B is reflected across the border pattern's vertical reflection axis to the point C. We know that if a translation vector takes points in one direction it will also work in the opposite direction, and so the point C can be translated downward to D; but D is also the image of A when reflected over the vertical reflection axis when extended onto the wallpaper pattern.

We thus know that a border pattern will have vertical reflection if and only if there are reflection axes within the parent wallpaper pattern running perpendicular to the direction of the border pattern. Since there are no reflection axes in a p1 pattern, we know that a p1 contains no pm11 border patterns.

Note that in lemmas 1-4 above, the point A may be any distance from the border pattern. As stated, A can be mapped into the border pattern by some integer multiple of the tiling translation vector. This vector will remain a valid translation within the tiled wallpaper pattern.

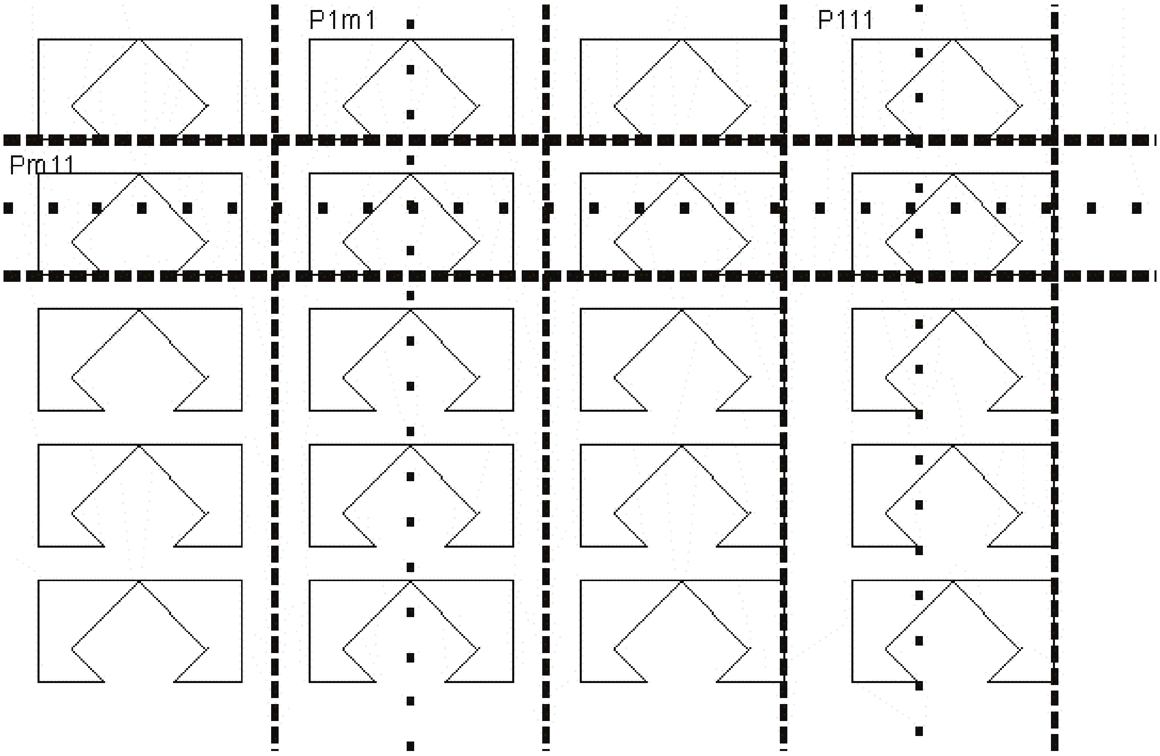
In summary, the only type of border pattern that can be found in a p1 wallpaper pattern is the p111. The only fact still to be proven is that translation vectors must extend to the entire pattern if they translate a border pattern (Appendix 1).

We now have a set of lemmas proving that no isometry can exist in a border pattern unless it extends to the parent wallpaper pattern. For now, I ask that the reader accepts the claim that for any translation vector in a wallpaper pattern one can find a translation vector perpendicular to the original. Clearly, if this were not true, problems would quickly arise in the discussion of the p1 and other patterns that have no inherent isometry structure guaranteeing pairs of perpendicular translation vectors.

The pm pattern

The pm pattern has only one direction of reflection, so by lemma 2 there will be no half turn and by lemma 3 there will be no glide reflection in any of the border patterns located within the pm pattern. This leaves us with the possibility of p111, p1m1, and pm11. Below, our example of a pm pattern is shown to contain each of these border patterns. One can see that each border pattern will also tile this pm, by translating it by the vector defining its own width.

Figure 6: The border patterns within the pm



The p1m1 and p111 patterns are shown running vertically. The center line of the p1m1 must lie on the reflection axis. In this case, vertical border patterns will be p111's except for the rare occasions when the center line falls on a reflection axis. The p111 in this pattern, as in any wallpaper pattern, can be found in an infinite number of directions, as long as there exists a pair of perpendicular translation vectors in those directions. I do not show any pairs of perpendicular translation vectors in this pattern because they are created by the existence of reflection.

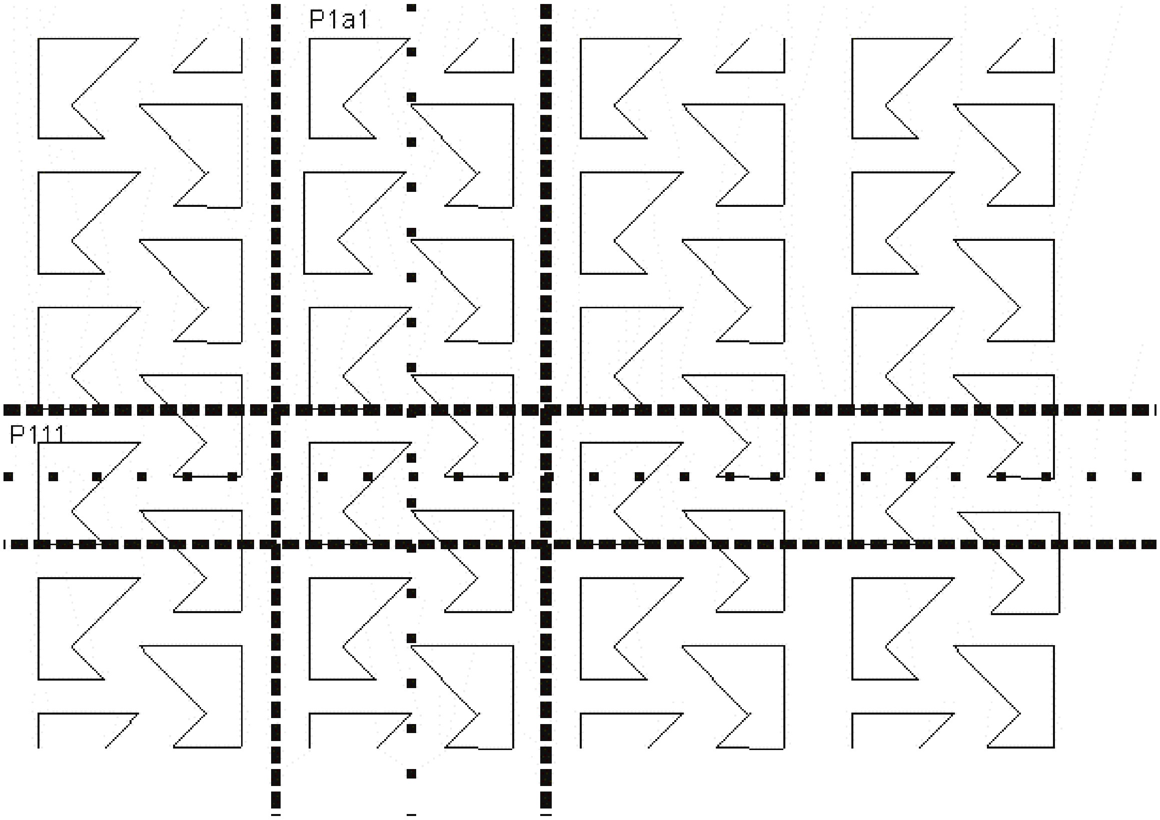
The pm is the first pattern in which we see a common form. The pm11 and p1m1 will run perpendicular whenever the p1m1 pattern exists, due to the necessity of reflection axes in perpendicular directions. Frequently, p111 will also run perpendicular to the pm11. The only patterns that may lie in a direction perpendicular to that of the p1m1 are the pm11, pma2 and pmm2, due to their perpendicular reflection. The pm11 must lie horizontal in this pm pattern, as it must be perpendicular to the reflection axis. All horizontal border patterns in this pm pattern will be pm11's. This is due to lemma 4. This is the only direction with translation but without a p111 pattern. I will not show any slanted p111 patterns, though they may exist, since it is not needed.

Note also that when two border patterns are perpendicular, as are the p1m1 and pm11 in the pm, or in figures 1a and 1b, the translation vector which defines the width of one border pattern will be the translation vector of the other border pattern and vice versa.

The pg pattern

We know that the pg pattern contains only glide reflection in one direction. By lemmas 1 and 4, we know that since there is no reflection within this pattern there will be no vertical or horizontal reflection in the border patterns it contains. Lemma 2 tells us that there will be no border patterns with half turn, since there are no rotation points within a pg. We are thus left with p111 and p1a1 as the only possible border patterns that the pg can contain. The following diagram shows the existence of these two border patterns, in our example of a pg.

Figure 7: The border patterns within the pg

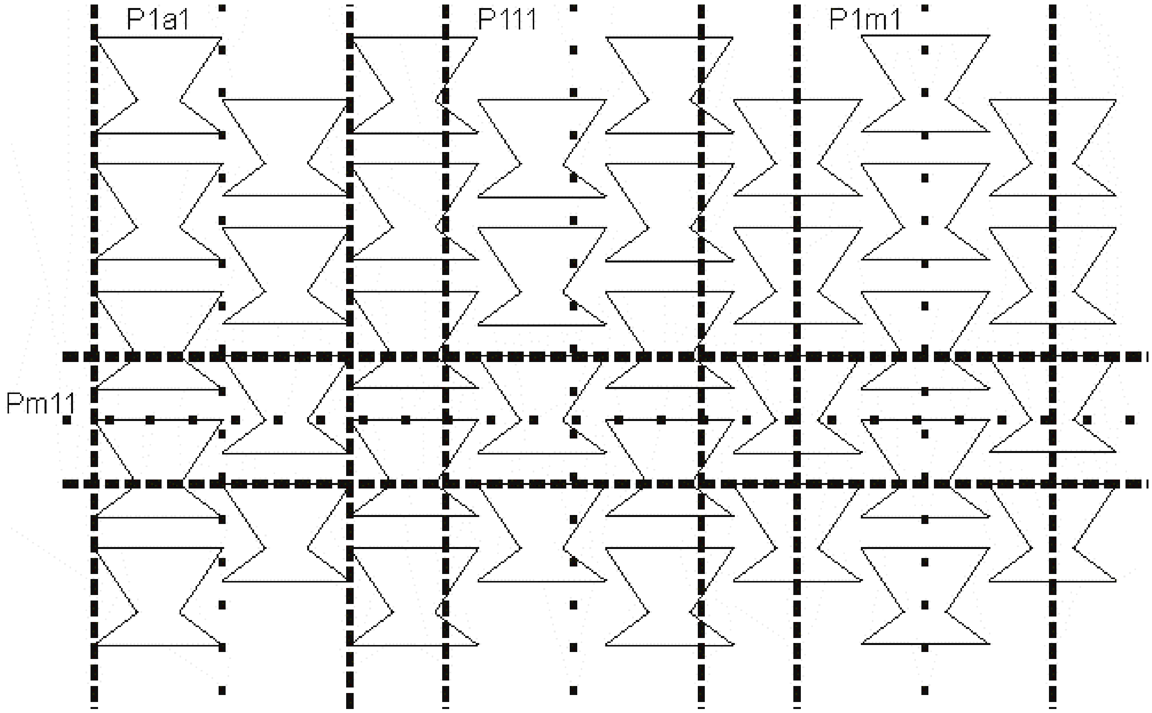


Here, the p1a1 contains a glide reflection axis on its center line, while the p111 lies perpendicular to all glide reflection lines and thus contains no isometries. The p111 could also have been created in any direction where there existed perpendicular translation vectors, including parallel to the p1a1 pattern, as long as the center line did not fall on any glide reflection axis. In this situation, the p1a1 is to the vertical p111 as the p1m1 is to the vertical p111 in the previous pm pattern.

The cm pattern

The cm pattern contains reflection in one direction and parallel glide reflection. Since it contains no rotation points it will contain no p112, pma2 or pmm2 border patterns, by lemma 2.

Figure 8: The border patterns within the cm

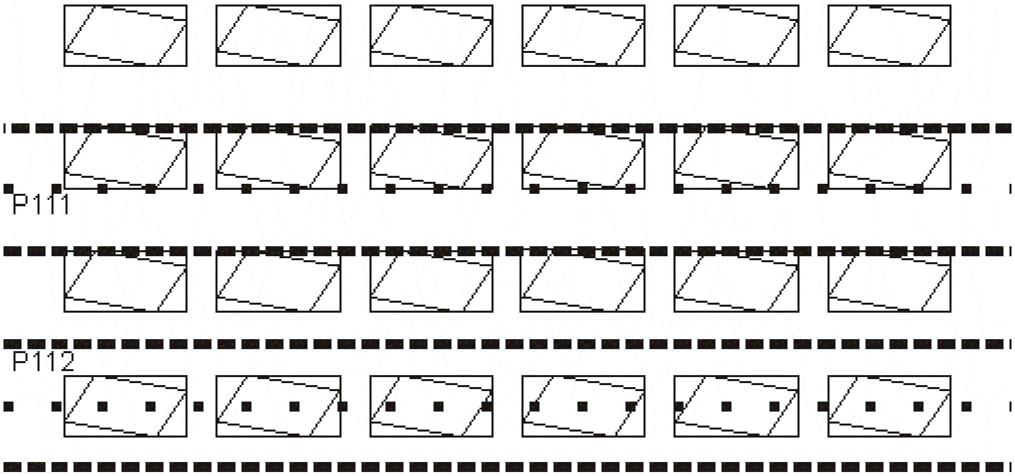


Here, the cm pattern contains reflection, as well as glide reflection, axes running vertically, and so the p1a1 has a center line on the vertical glide reflection axis and the p1m1 has a center line on the vertical reflection axis. The pm11 of course must be perpendicular to the p1m1, since it requires vertical reflection and the cm contains reflection in only one direction. Clearly, the center line in the p111 contains no isometries. Similar to the pm, the horizontal direction contains only pm11's and so it will be the only direction with perpendicular translation that contains no p111 patterns, because of vertical reflection. Similar to the pg and pm, the cm has obvious p111 and so I have not shown any slanted examples.

The p2 pattern

Since the p2 pattern contains only 180^{o} rotation points, and no reflection or glide reflection, no border patterns other than the p112 and p111 are possible, by lemmas 1, 3 and 4. These two border patterns are shown in the diagram below.

Figure 9: The border patterns within the p2

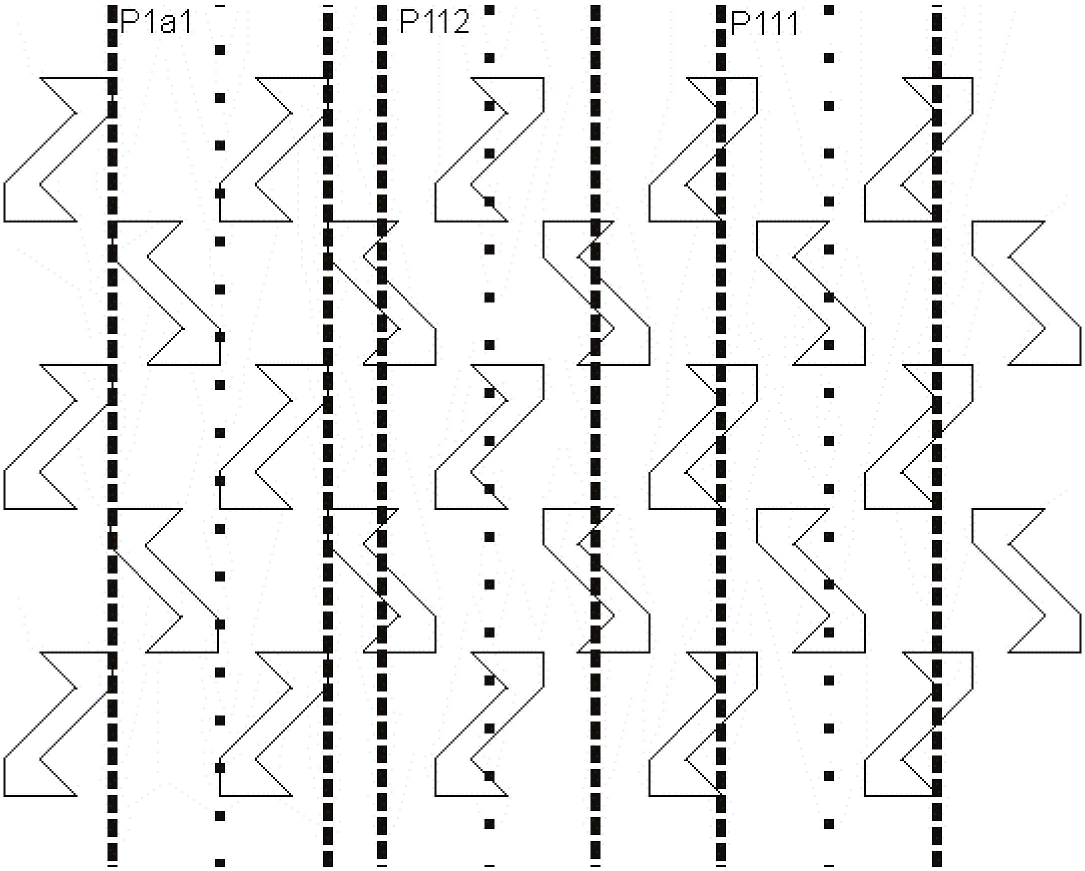


Since the p2 pattern has only rotation, the classification of any border pattern within the p2 depends on whether or not the center line contains rotation points. The p111's center line contains no rotation points, whereas the p112's center line contains infinitely many. While in figure 9 the tiling of the p2 pattern by p111 and p112 patterns is possible due to the particular p2's structure, in the general case such tilings rely on the existence of pairs of perpendicular translations.

The pgg pattern

Since the pgg pattern contains 180° rotation points and glide reflection axes but has no reflection, it cannot be tiled by pm11, p1m1, or pmm2. The p111, p112 and p1a1 patterns tiling the pgg are shown below and we are left with the question of the pma2, which does not exist within the pgg despite the existence of both glide reflection and rotation points. The non-existence of the pma2 is shown by lemma 4 (since the pma2 also has vertical reflection and the pgg has no reflection at all) and is also apparent since no glide reflection axes in a pgg pattern contain the rotation points that exist on the center line of a pma2.

Figure 10a: The border patterns within the pgg



Since all glide reflection axis in the pgg are perpendicular to other isometries of the same type, all patterns that are found in the pgg pattern can be found in two directions. For example, in the pgg pattern above, p111, p112, and p1a1 border patterns are identified running vertically. Below the same border pattern types are shown running horizontally.

Figure 10b: The horizontal border patterns within the same pgg

In addition to existing in the vertical and horizontal direction, the p111 and p112 patterns will tile in infinitely many diagonal directions. Of course this is always dependent on the existence of perpendicular translation vectors in those directions.

The pmm pattern

The pmm pattern contains two directions of reflection with rotation points lying on the intersection of any two perpendicular reflection axes. Clearly, the absence of glide reflection implies the absence of p1a1 or pma2 patterns, by lemma 3. Also, there will be no p1m1 pattern inside a pmm. It should be clear that a wallpaper pattern will contain a p1m1 pattern if and only if it contains a reflection axis that does not connect a set of rotation points. The existence of rotation points on the necessary reflection axis which is its center line will turn any p1m1 into a pmm2. Below I provide examples of horizontal pmm2 and pm11 that tile the pmm, followed by the skewed p111 and p112 patterns.

Figure 11a: The pm11 and pmm2 patterns within the pmm

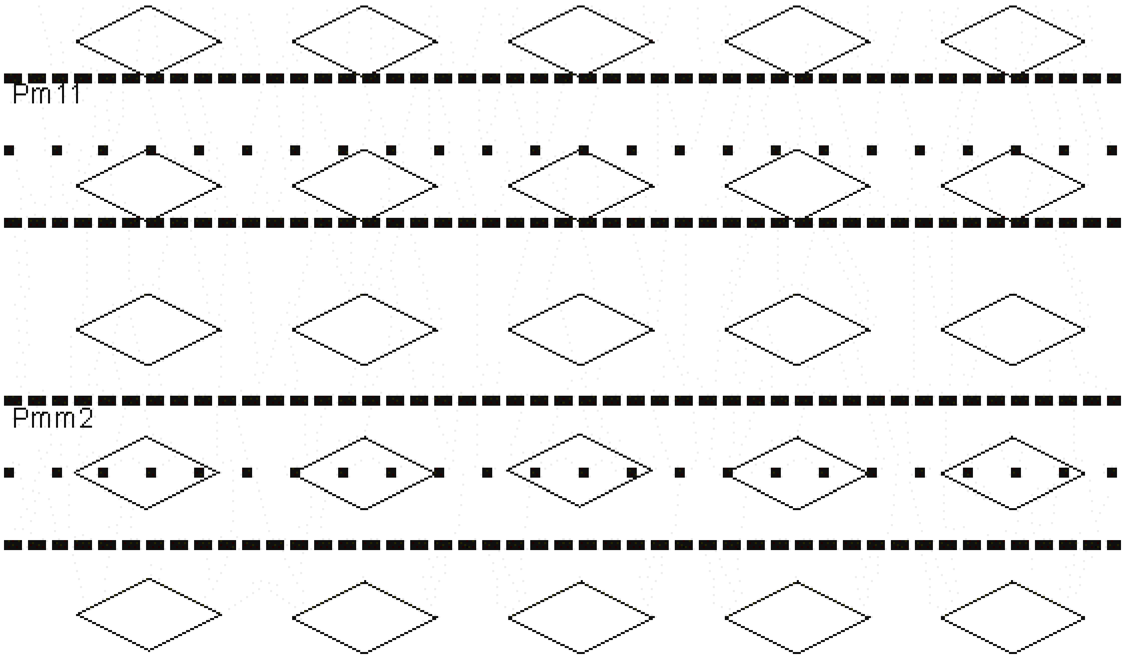
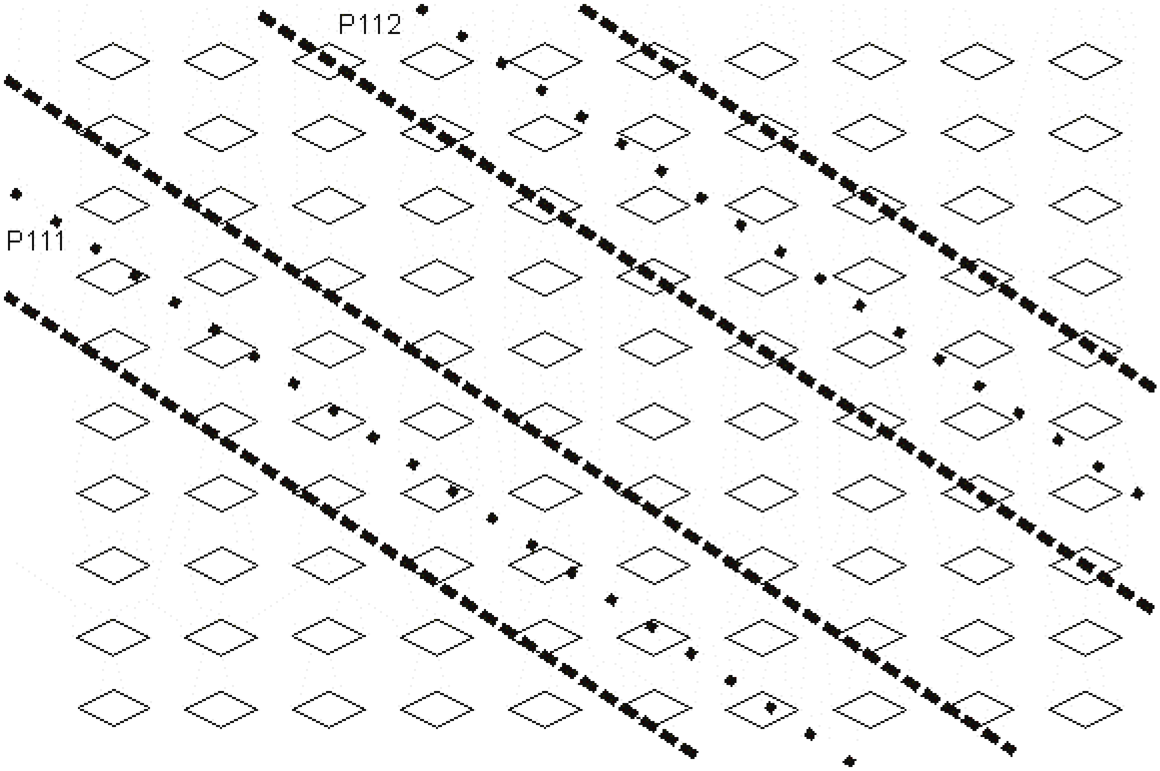


Figure 11b: The p111 and p112 border patterns within the pmm

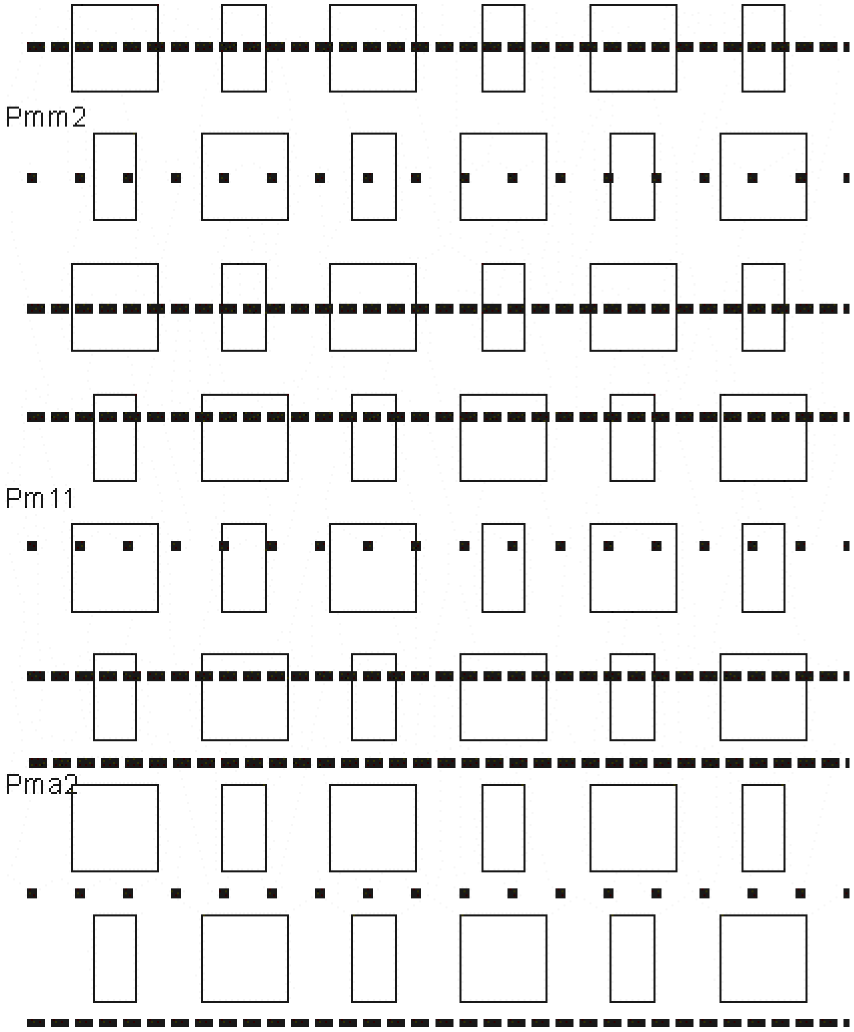


Clearly, since the p111 and p112 patterns have no vertical reflection, they may not run perpendicular to any reflection axes, and so they cannot be vertical or horizontal in the pmm, which contains both vertical and horizontal reflection axes. The existence of these slanted patterns, which are shown in figure 11b, and their ability to tile the pmm, relies on the existence of pairs of perpendicular translation vectors. The center line of the p112 must contain rotation points as shown in figure 11b without lying on, or crossing at 90^o angles, any reflection axes. Both of these patterns could exist in infinitely many directions, as long as they are not perpendicular to any reflection axis.

The cmm pattern

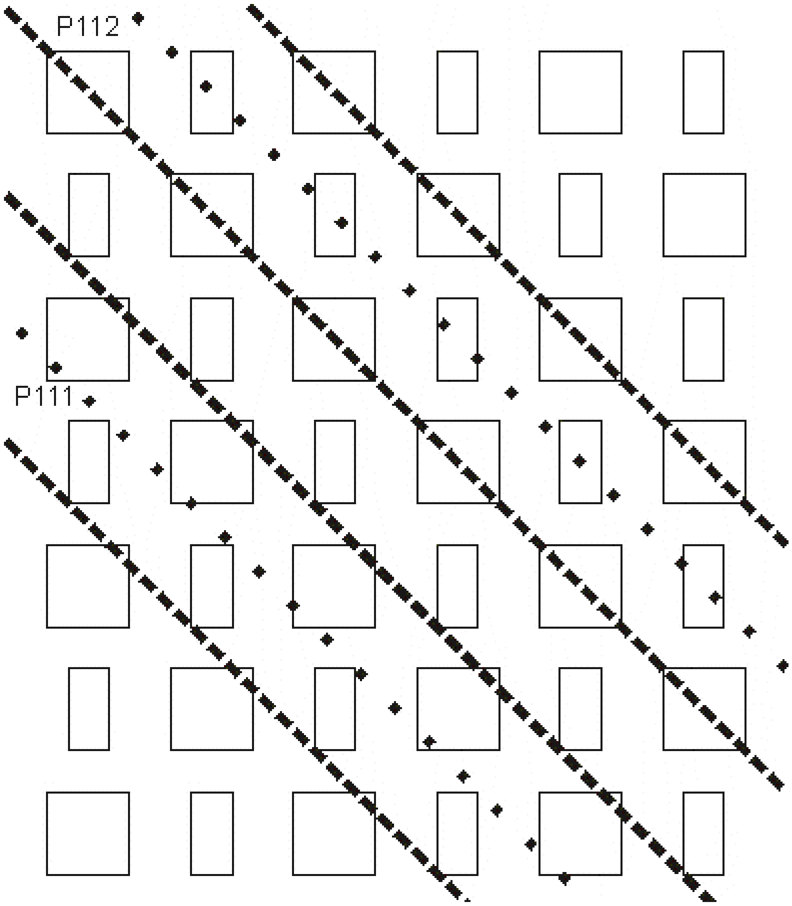
The cmm is the first wallpaper pattern that has reflection, glide reflection, and half turn. It may be tiled by p111, p112, pm11, pma2, and pmm2 border patterns. This pattern contains reflection and glide reflection axes, which run through rotation points, and so the pmm2 and pma2 can clearly tile the cmm, while the p1m1 and p1a1 cannot. Also, whenever the border pattern’s center line runs parallel to but does not coincide with a reflection or glide reflection axis, a pm11 is created. These three types are shown below.

Figure 12a: The horizontal border patterns within the cmm



The cmm pattern is another example that has sets of parallel border patterns. Since the pm11, pmm2 and pma2 all require vertical reflection, they can be found in this example in the horizontal or vertical direction. Patterns that are found running horizontally or vertically in this pattern will be mostly pm11's, with the occasional pmm2 and pma2 appearing when the center line coincides with a reflection or glide reflection axis, respectively.

Figure 12b: The slanted border patterns within the cmm

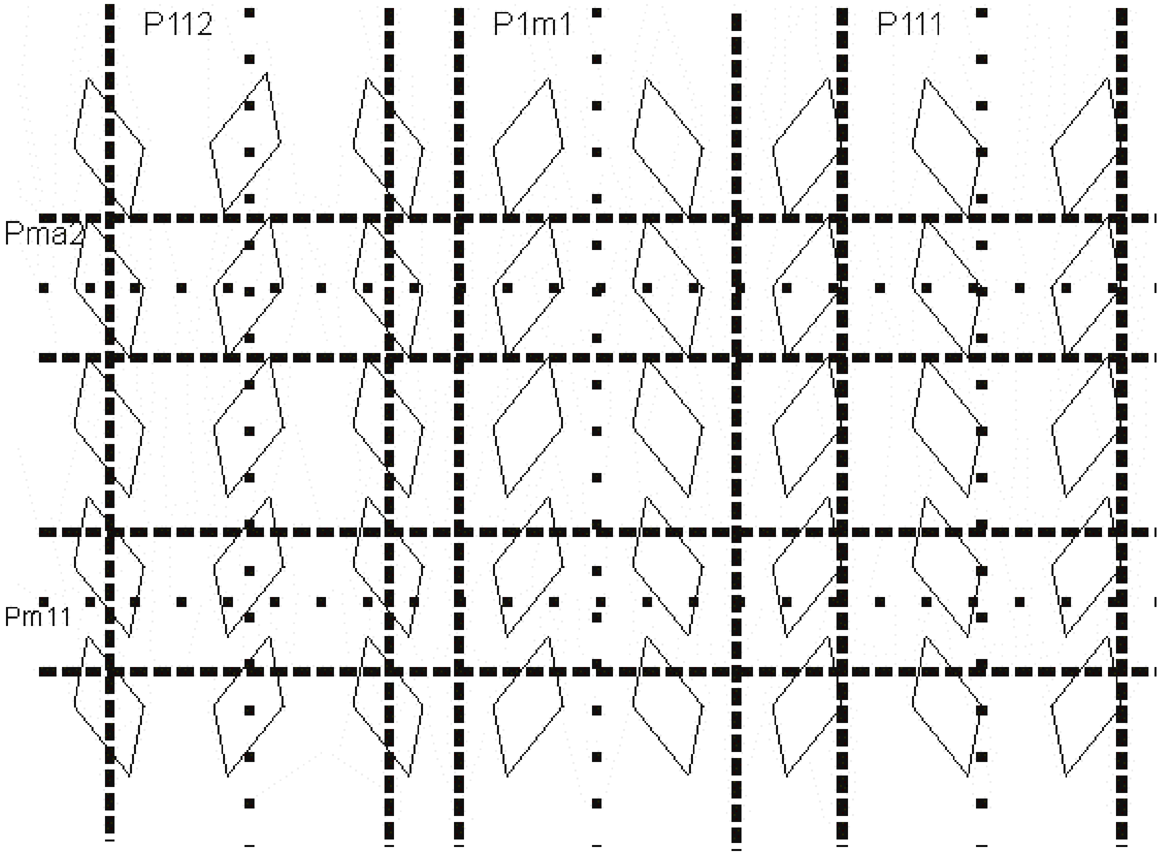


The two directions of reflection necessitate slanted p111 and p112 patterns, which do not have vertical reflection. As in many other wallpaper patterns, these two types can be found in infinitely many diagonal directions, provided that perpendicular translation vectors exist in those directions.

The pmg pattern

The pmg pattern has reflection, glide reflection, and half turn. The pmg contains no pmm2 pattern since it has no reflection axes with rotation points on it. Also, since all glide reflection axes contain rotation points, there can be no p1a1 border patterns. The others, namely the p111, p112, p1m1, pm11 and pma2 patterns, are shown below.

Figure 13: The border patterns within the pmg

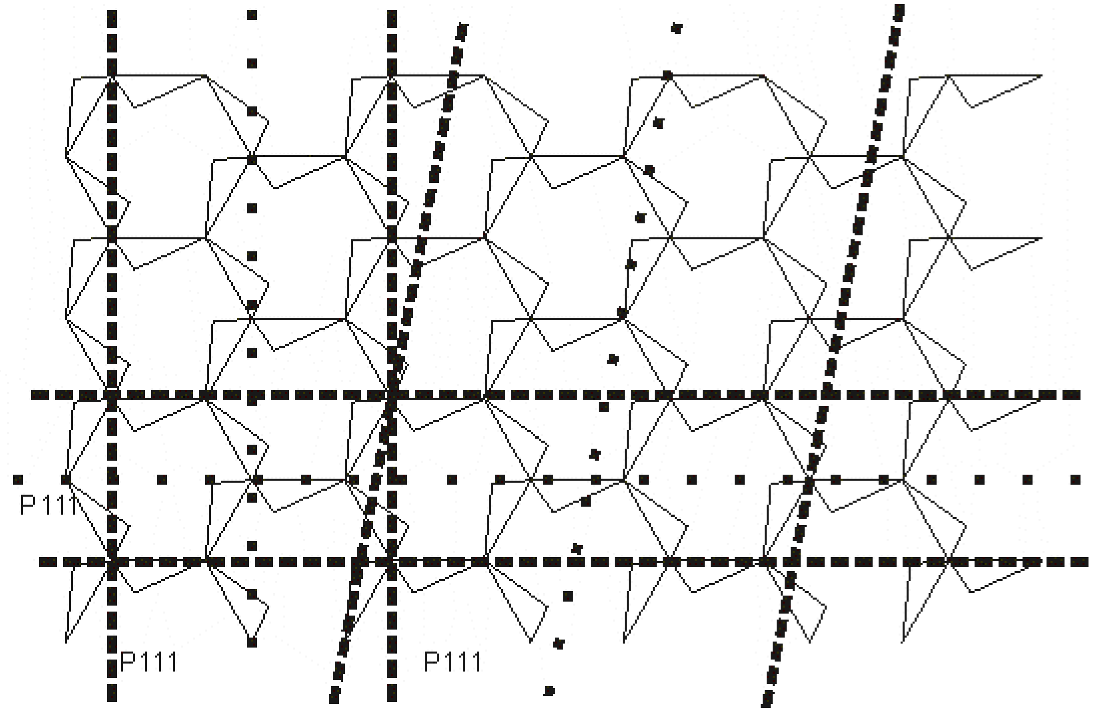


The fact that no isometry exists in both directions causes all of the border patterns (with the possible exception of slanted p111 and p112) that may tile the pmg to exist in only one direction. Both the pm11 and the pma2 must be perpendicular to reflection axes and since the pmg has only one direction of reflection, these two patterns can only run in the horizontal direction. Similarly, the p1m1 must run in the direction of the reflection axis and so it can only run vertically in this pattern. The p112 and p111 can run in any direction, as long they are not perpendicular to any reflection axis. These two patterns are shown here as shifted versions of the p1m1, though both could also be found in infinitely many non-horizontal directions as long as there exist perpendicular translations in those directions.

The p3 pattern

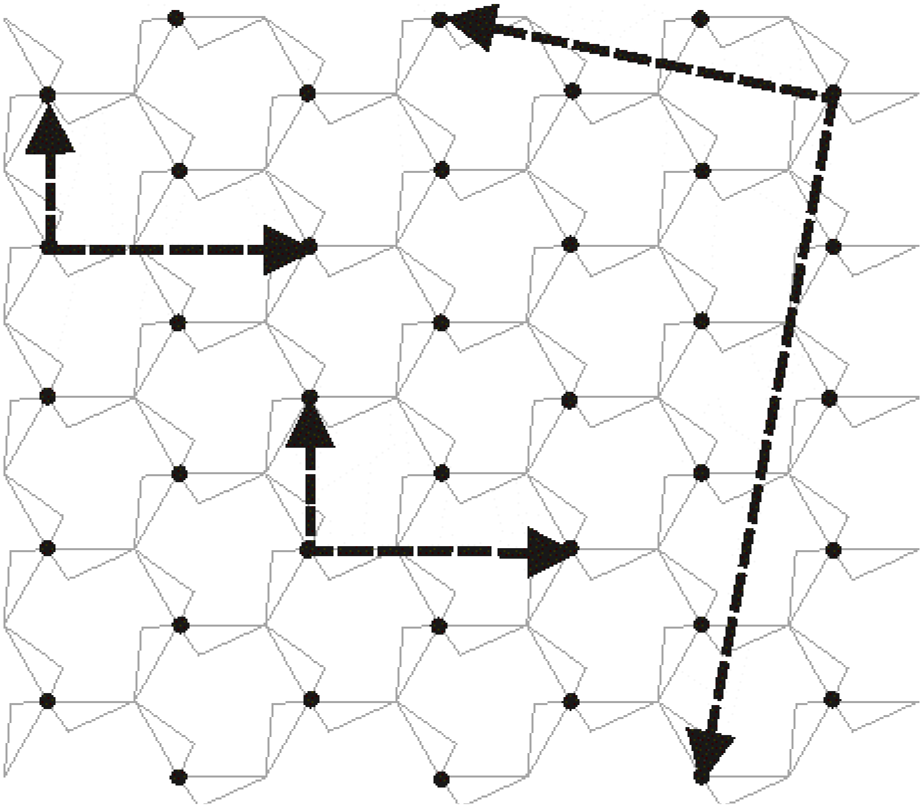
Since the p3 pattern has no reflection, no glide reflection, and no half turn, by lemmas 1-4 there are no border patterns other than the p111 found within it. Note that even though threefold rotation points exist, this isometry does not concern us while we are looking for border patterns.

Figure 14a: The border patterns within the p3



Here the only type of border pattern is a p111. I show three examples running through the rotation points, but because the rotations are 120° they do not apply to a border pattern anyway. Like the p1 and p2 patterns, the p3 will have p111 border patterns in infinitely many directions, in fact – and since there is no reflection -- in all directions with perpendicular translations. But this time pairs of perpendicular translations are generated by the theefold rotations: the p3 grid below shows the various pairs of translation vectors that create different p111 patterns.

Figure 14b: The grid of rotation points in a pattern with threefold rotation

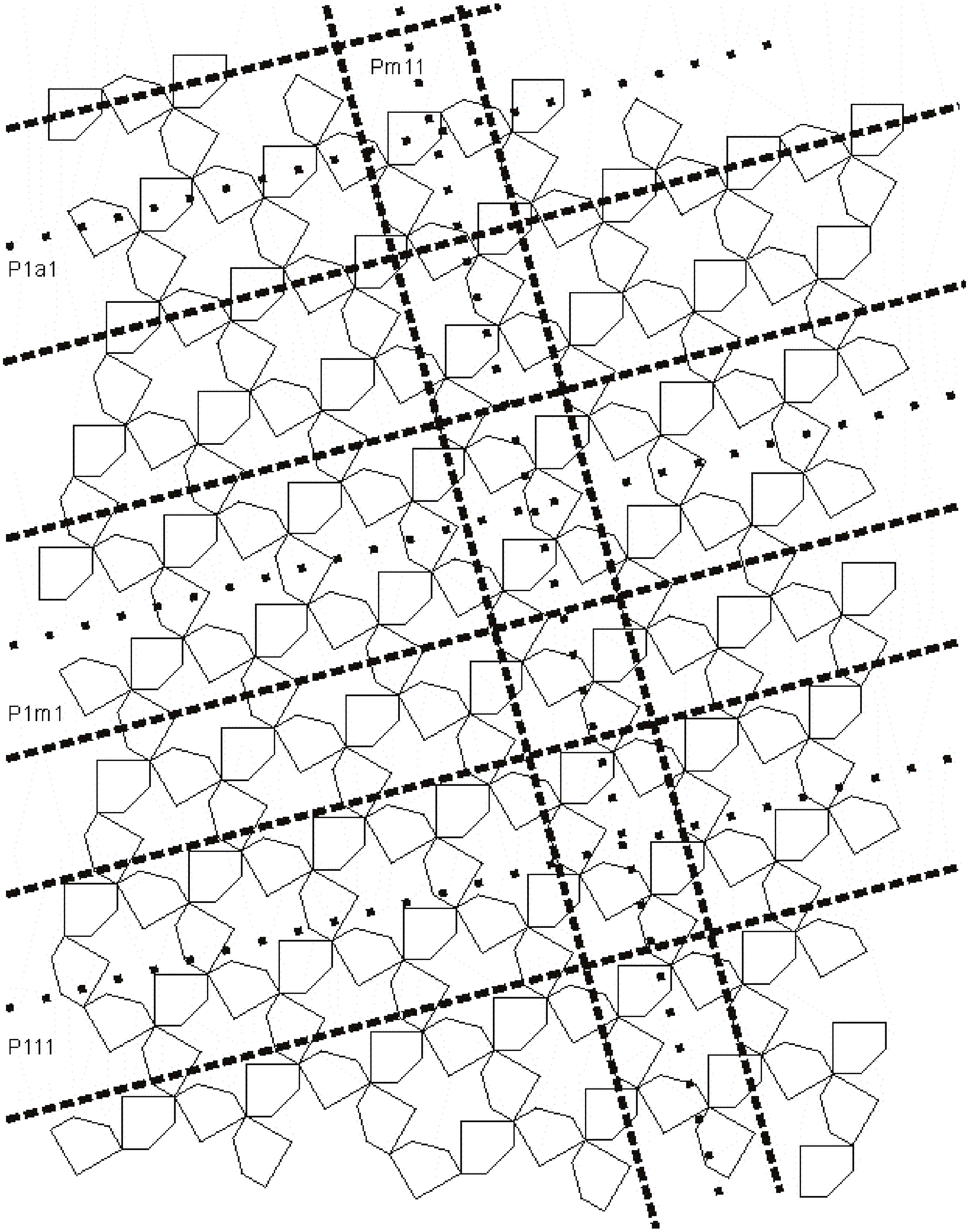


This diagram shows the grid structure that exists in the p3, p31m, p3m1, p6 and p6m patterns. Of course there are infinitely many pairs of perpendicular translation vectors in this pattern. I have chosen to show the three examples that are responsible for creating the set of p111 patterns displayed in figure 14a. Two of these utilize the shortest possible pair of translation vectors. It should be clear that all of these vector lengths work not only at any placement on the grid pattern, but also when rotated by 120°; see also Appendix 2.

The p31m pattern

The p31m contains reflection and glide reflection in three directions. It contains only the “useless” threefold rotation, and thus no half turn points. Lemma 2 tells us that there will be no p112, pmm2, or pma2 border patterns; the others, the pm11, p1a1, p1m1 and p111, are shown below.

Figure 15: The border patterns within the p31m

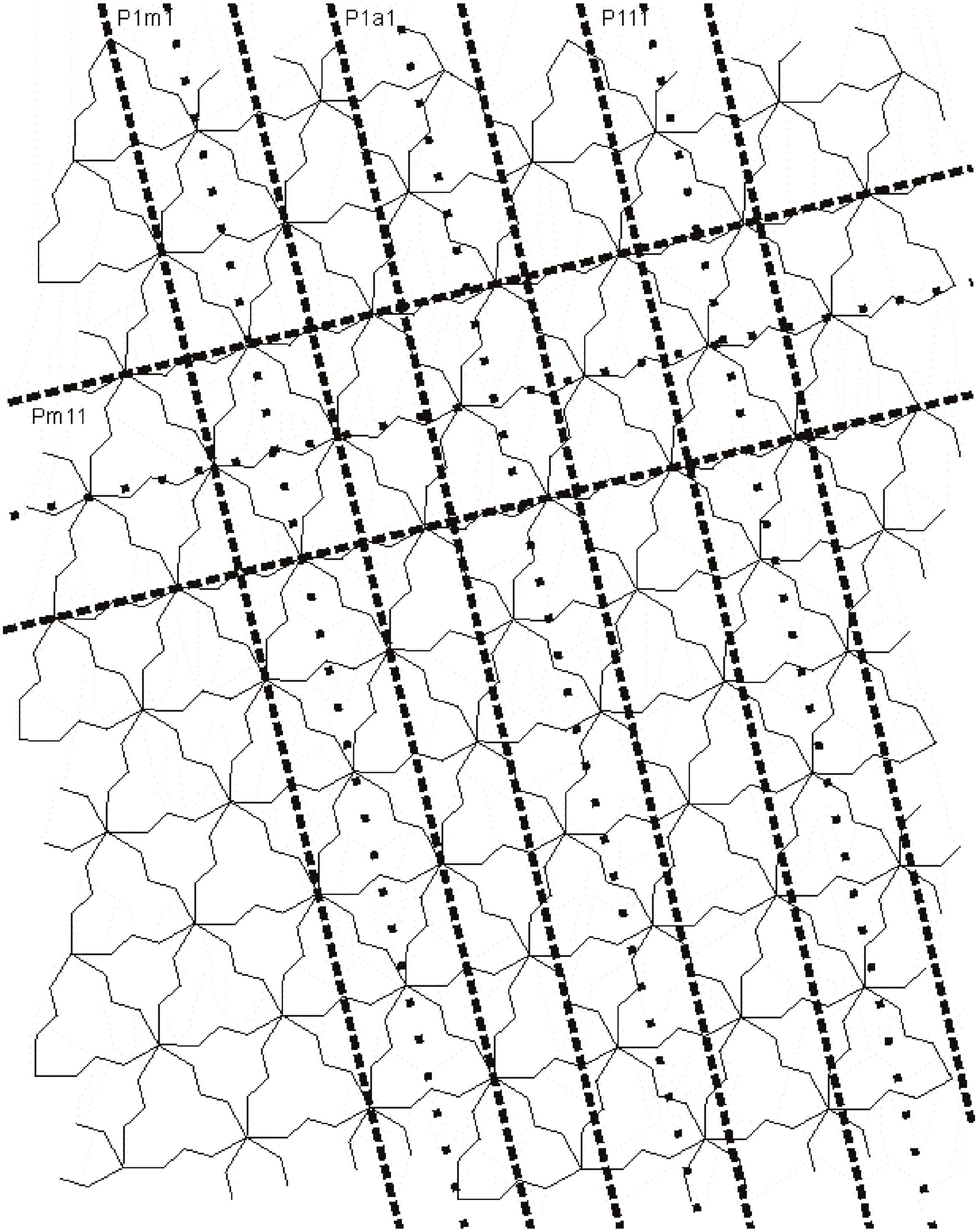


The p111, p1m1 and p1a1 patterns here are shown running parallel. They need not be parallel to one another, but each may run in any of the same three directions of the axes in the pattern, and so they could be shown running parallel in any of these three directions. Again, the p111 is the only one of these three which can be found in infinitely many directions, because its structure does not rely on the center line falling on a reflection or glide reflection axis. Of course the pm11 pattern lies perpendicular to the p1m1 since it requires vertical reflection. Like the other three border patterns, the pm11 can be found in any of three directions, perpendicular to any of the three directions of reflection.

The p3m1 pattern

The difference between the p31m and the p3m1 pattern is the placement of the threefold rotation points that are unimportant to us. The set of tiling border patterns and even their arrangement is identical in the p31m and p3m1. The only difference between the two sets of border patterns is that the translation vectors are reversed. In the p31m, the shortest translation vector defines the width of the pm11 (figure 15), while in the p3m1 the shortest vector runs perpendicular to the reflection axis and so the pm11 pattern will be the widest (figure 16).

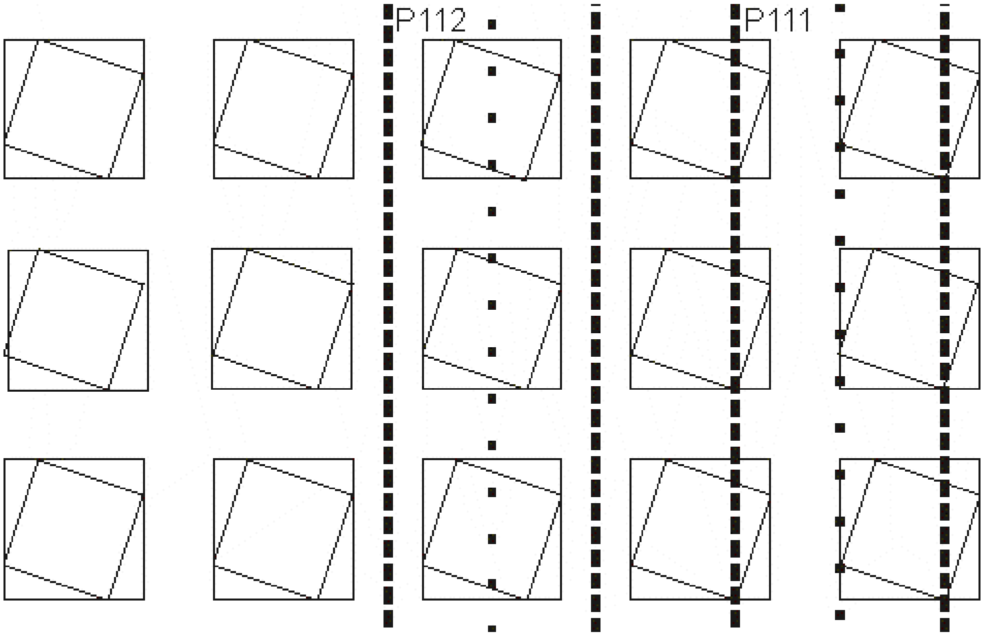
Figure 16: The border patterns within the p3m1



The p4 pattern

Similar to the case of the p2 pattern, the border patterns within a p4 pattern are easy to find. Lemmas 1-4 tell us that no border patterns will tile the p4 other than the p111 and p112, which are shown below.

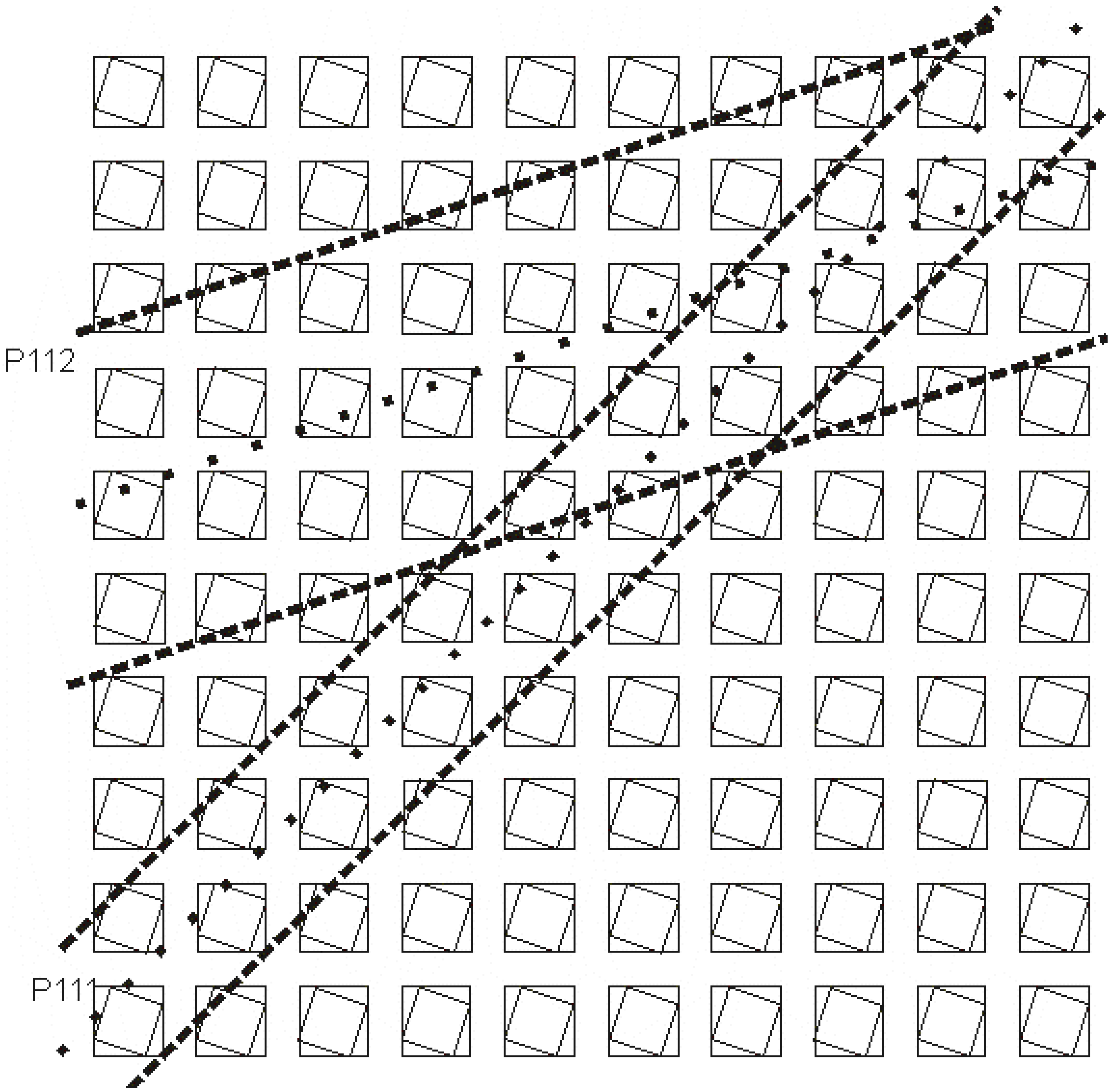
Figure 17a: The border patterns within the p4



Though the p111 and p112 are the only border patterns contained in the p4, it is, regardless, an interesting pattern in that both types of border patterns can be found in infinitely many directions. As always, the p111 patterns can be found in every direction that holds a p112, by simply shifting the pattern until the center line no longer contains rotation points. The p112 patterns are found in the rare cases when a center line crosses either two types of 180° rotation points or two types of 90° rotation points or, as in figure

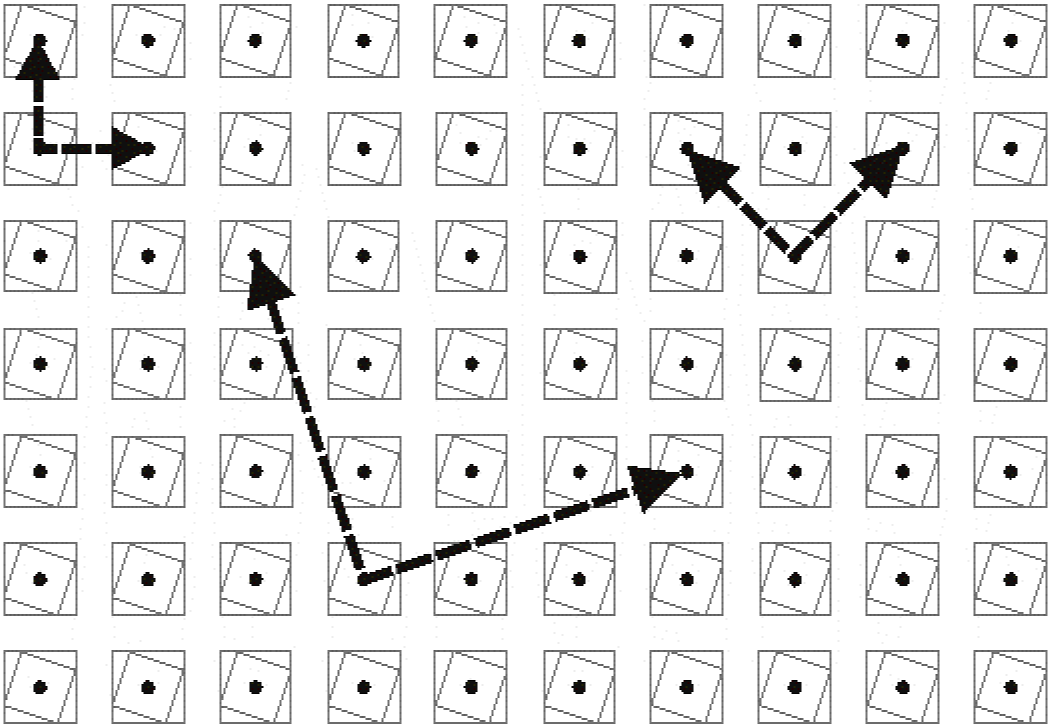
17a, one of each. Any set of two like rotation points will define a translation vector, and thus imply the existence of a p111; see also figure 17c and Appendix 2.

Figure 17b: The slanted border patterns within the p4



Here, p111 and p112 that use longer translation vectors are shown. These examples are wider and more difficult to see, but function exactly the same as the border patterns in figure 17a. Of course the p111 pattern will also exist in the direction of the p112, with the same width, and the p112 will also exist in the direction of the p111.

Figure 17c: The grid structure in the p4



The grid within the p4 is shown above as it was for the p3 earlier. Here, the fourfold points in the center of the squares are shown. I show the three sets of vectors that define the border patterns shown in figures 17a and 17b.

The p4g pattern

The p4g pattern has glide reflection, reflection and half turn; thus border patterns that can tile it are p111, p112, pm11, p1a1, pma2, and pmm2. No p1m1 patterns exist since all reflection axes contain half turn points. All other pattern types are shown below.

Figure 18a: The horizontal border patterns within the p4g

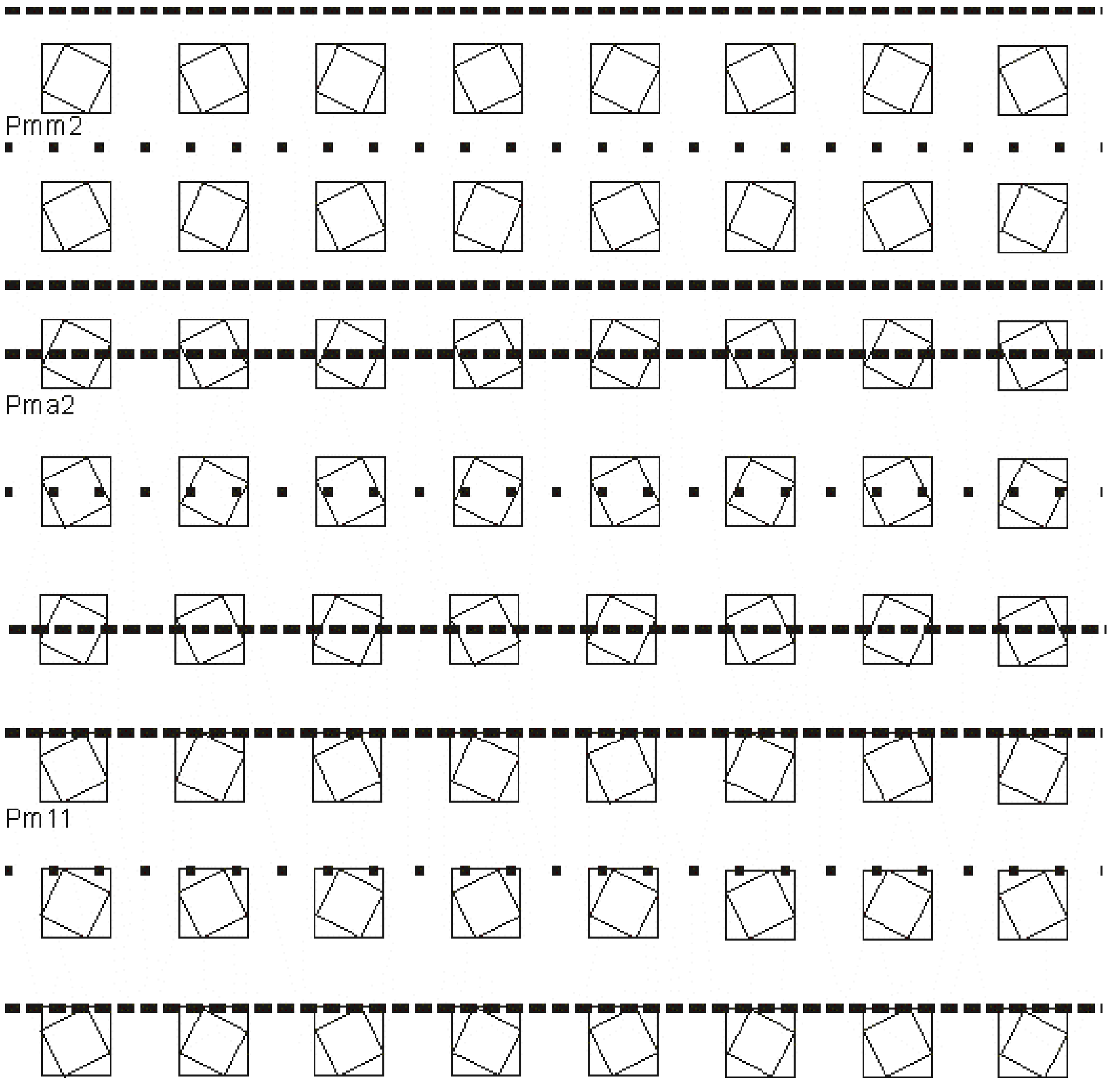
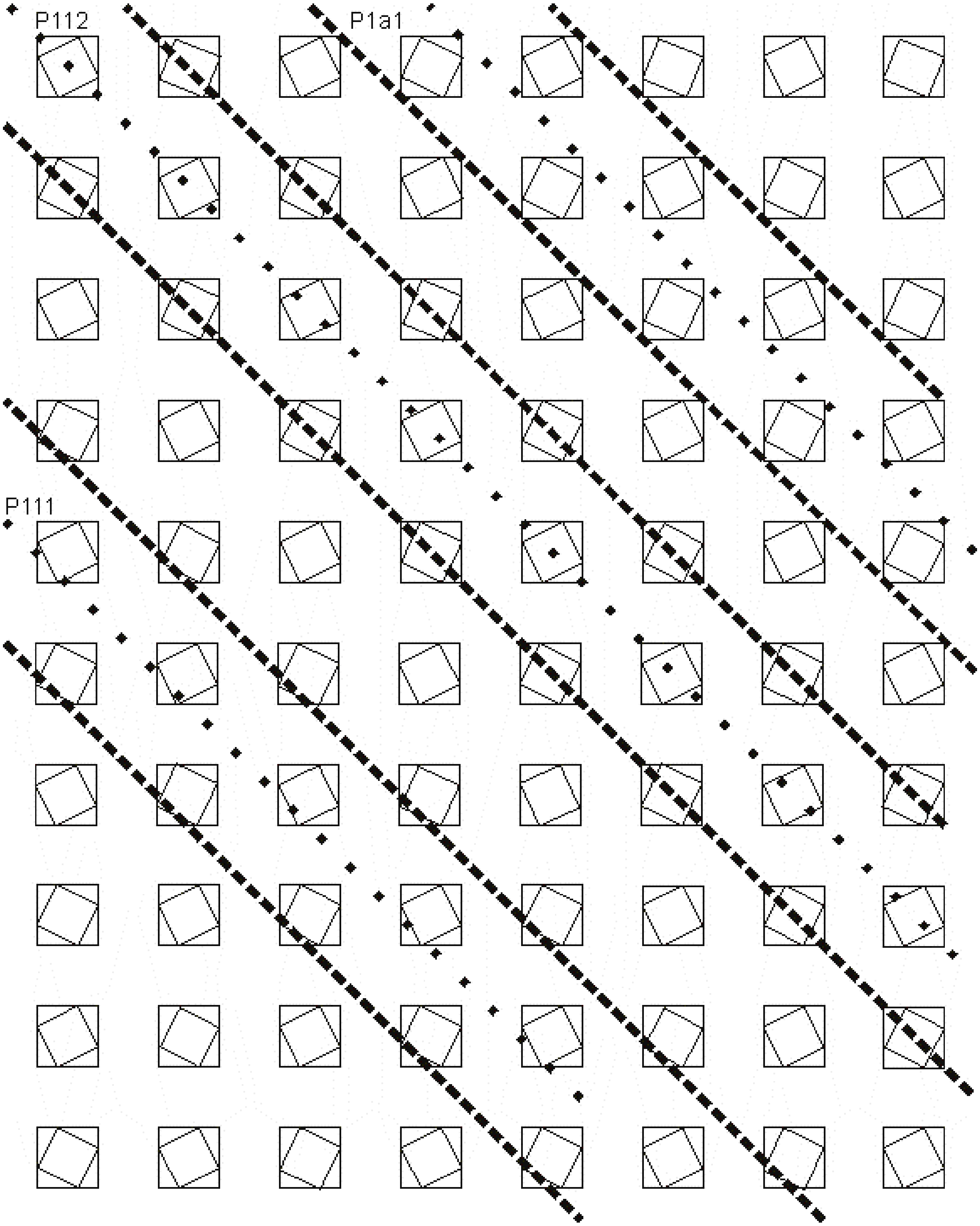


Figure 18b: The diagonal border patterns within the p4g



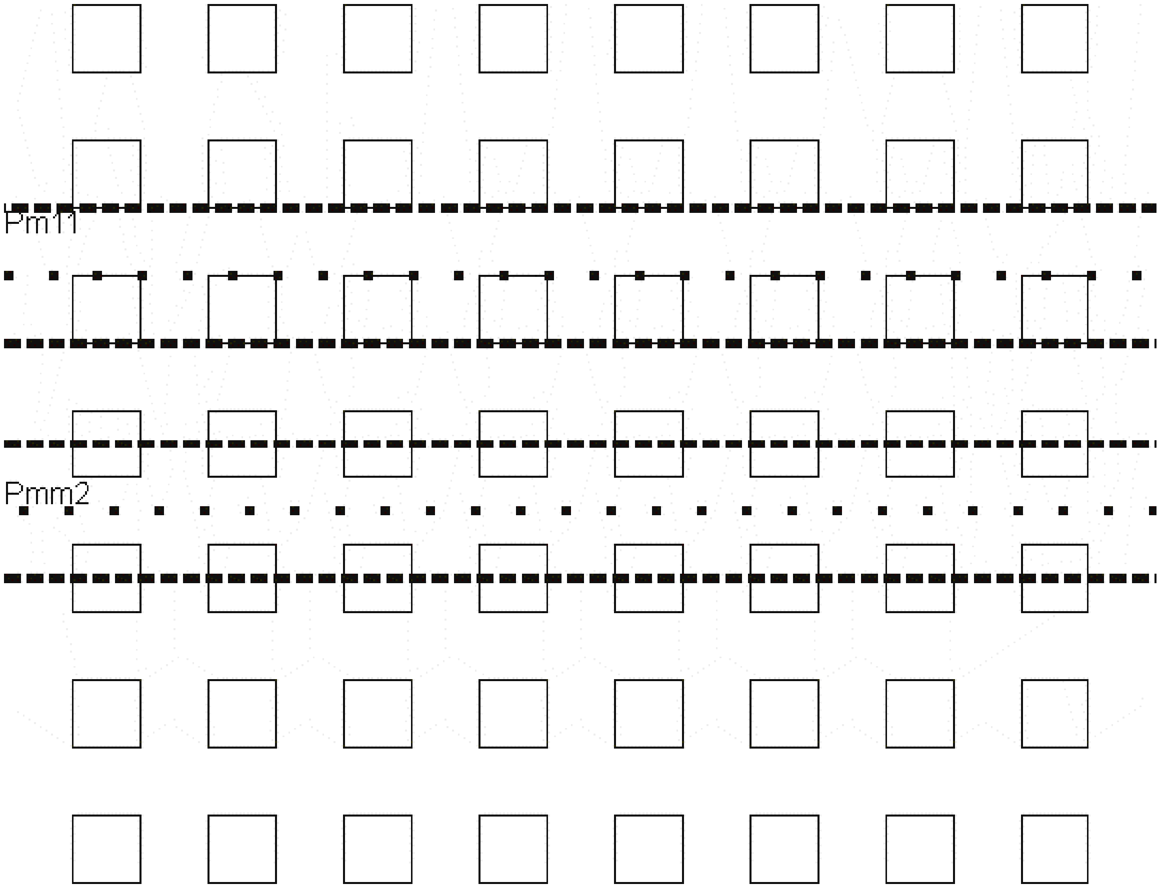
The border patterns within a p4g pattern run in two distinct directions. In figure 18a, the pma2, pm11, and pmm2 run horizontally, though they could also run vertically, since the p4g has 90^{o} degree rotation. The p111, p112, and p1a1, in figure 18b, are shown diagonally, though again they could be shown in the perpendicular diagonal direction, because of the same rotation.

The p4g pattern can be created by superimposing the cmm over the pgg. The pm11, pma2, and pmm2 patterns can thus tile the p4g in the same way in figure 18a as they tile the cmm in figure 12a. The p111, p112 and p1a1 will tile the p4g in the diagonal direction in a manner similar to the tilings of pgg in figures 10a and 10b. There also exist slanted p111 and p112 patterns in the p4g that I do not show here since they are guaranteed by the grid structure as in figures 17b and 17c.

The p4m pattern

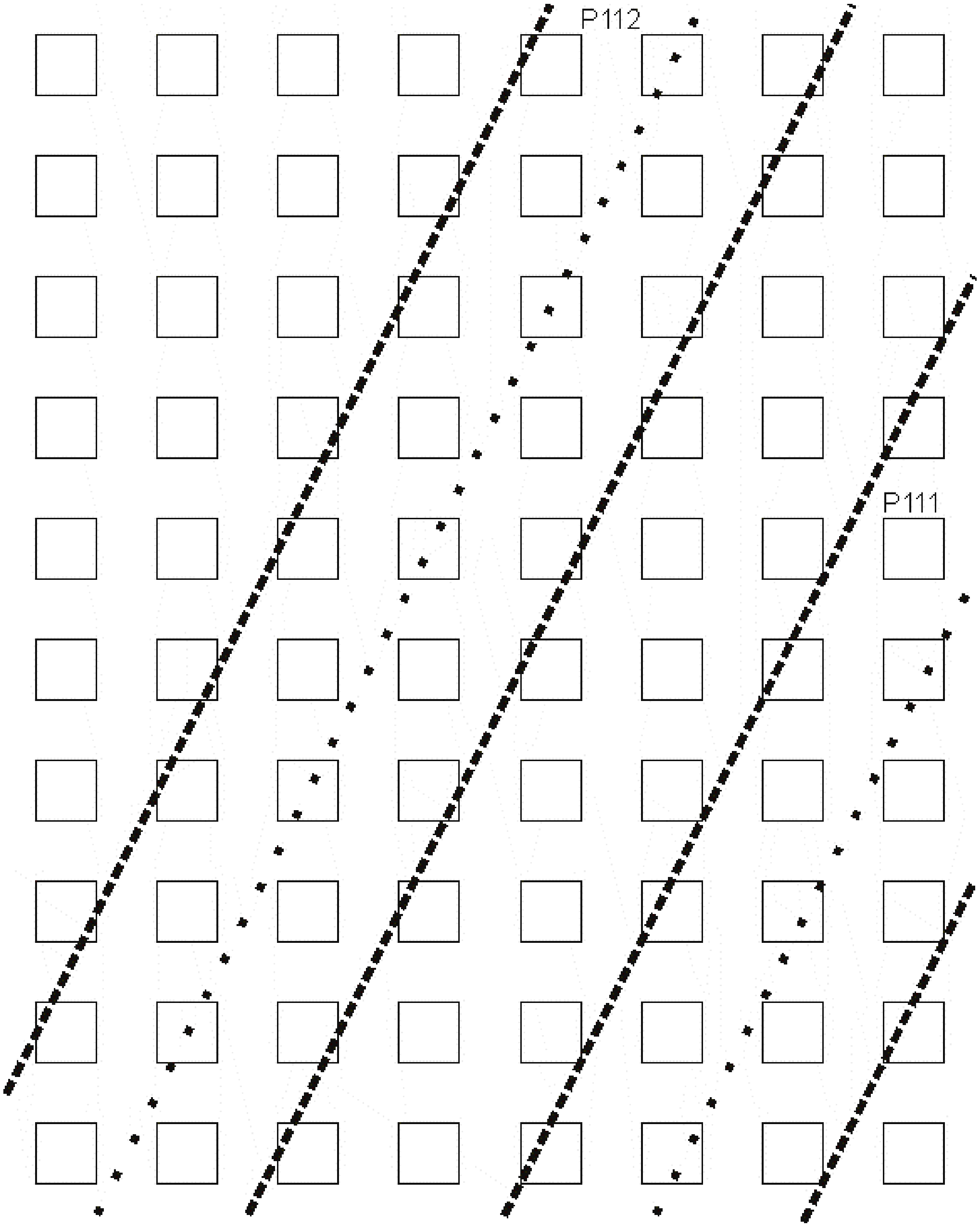
Like the p4g, the p4m has reflection, glide reflection and half turn. It also contains an almost identical set of border patterns: p111, p112, pm11, pma2 and pmm2. The p1m1 and p1a1 patterns are missing from the p4m for the same reason that the p1m1 is missing from the p4g: since all reflection and glide reflection axes contain rotation points it is impossible to find a p1m1 or p1a1 since both would require an axis containing no rotation points. Shown below, clearly the pm11, p112, p111, pma2 and pmm2 will tile the p4m pattern.

Figure 19a: The horizontal border patterns within the p4m



This figure shows the most obvious direction in which to locate border patterns. Only two of the five border patterns found in the p4m, however, can be found in this direction. Note that the 90° rotation will cause the same patterns to be found in the vertical direction. Most horizontal or vertical border patterns in the p4m will be pm11, and only those which have a center line on a reflection axis will be pmm2's.

Figure 19b: The slanted border patterns within the p4m



This direction contains mostly p111 patterns with p112's appearing wherever the center line passes through rotation points. The same grid structure as in the p4 allows for tiling the p4m, using the p111 and p112 patterns, in infinitely many directions.

Figure 19c: The diagonal border patterns within the p4m

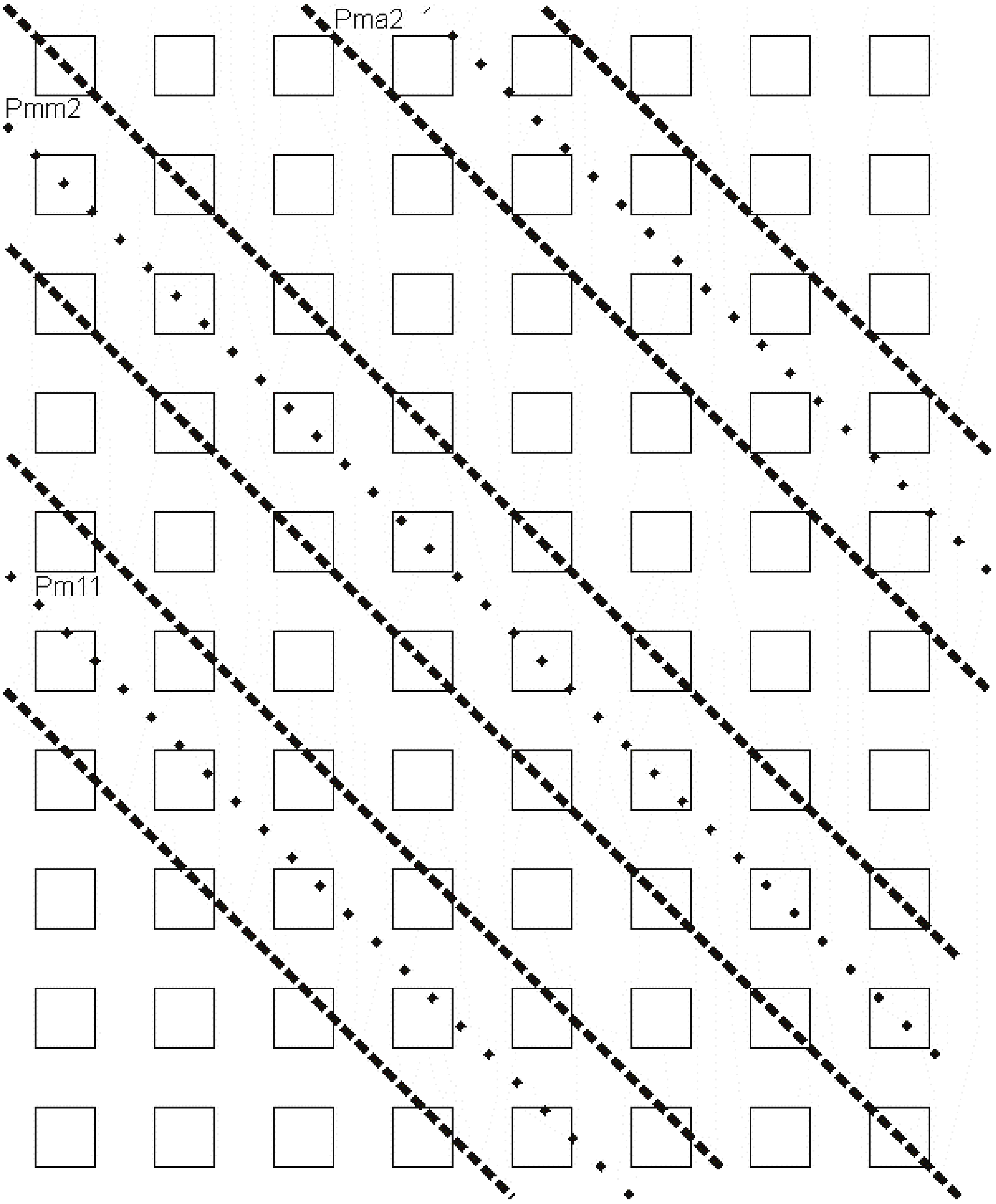
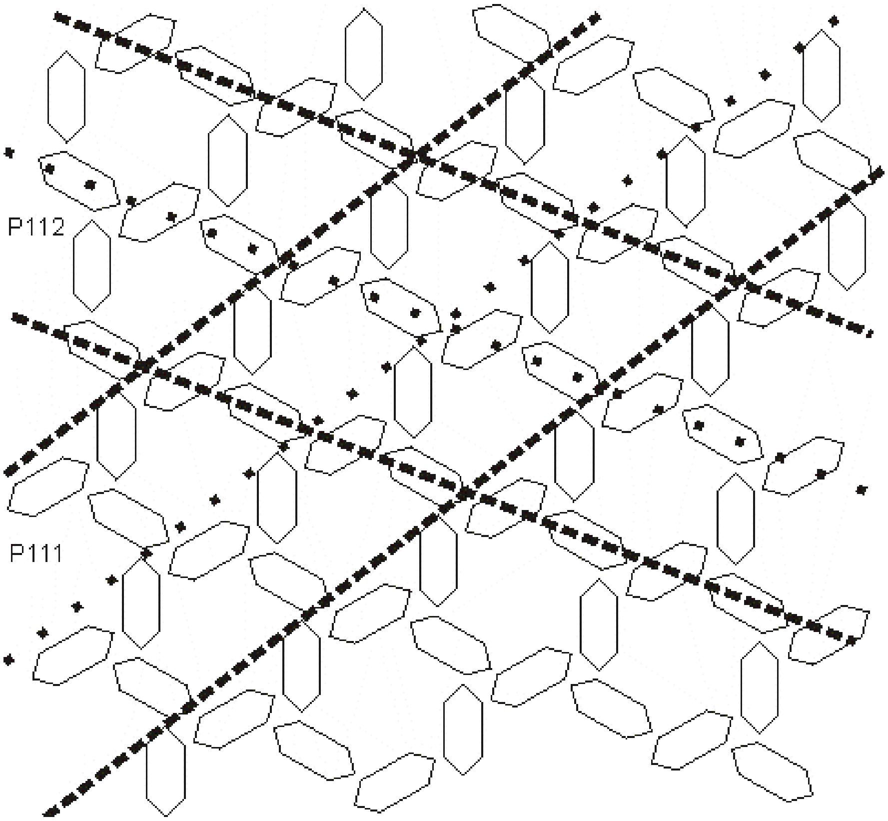


Figure 19c shows the pma2 pattern along with another way of finding the pm11 and the pmm2, which are both also shown in the horizontal direction in figure 19a. The pma2 can only be found in the diagonal direction, unlike the pmm2 and pm11. In this direction, the majority of border patterns will be pm11, but, on rare occasions, a center line will fall on either a reflection or glide reflection axis and a pmm2 or pma2 pattern, respectively, will be formed.

The p6 pattern

The p6 pattern is built from a series of 60°, 120° and 180° rotation points. The 60° and 180° rotations are half turn points and so by lemmas 1, 3 and 4 we have p111 and p112 border patterns, the same as in the p2. The p6 however has a grid structure created by sixfold centers and similar to that of the p3: this no longer necessitates the assumption of perpendicular translation.

Figure 20: The border patterns within the p6

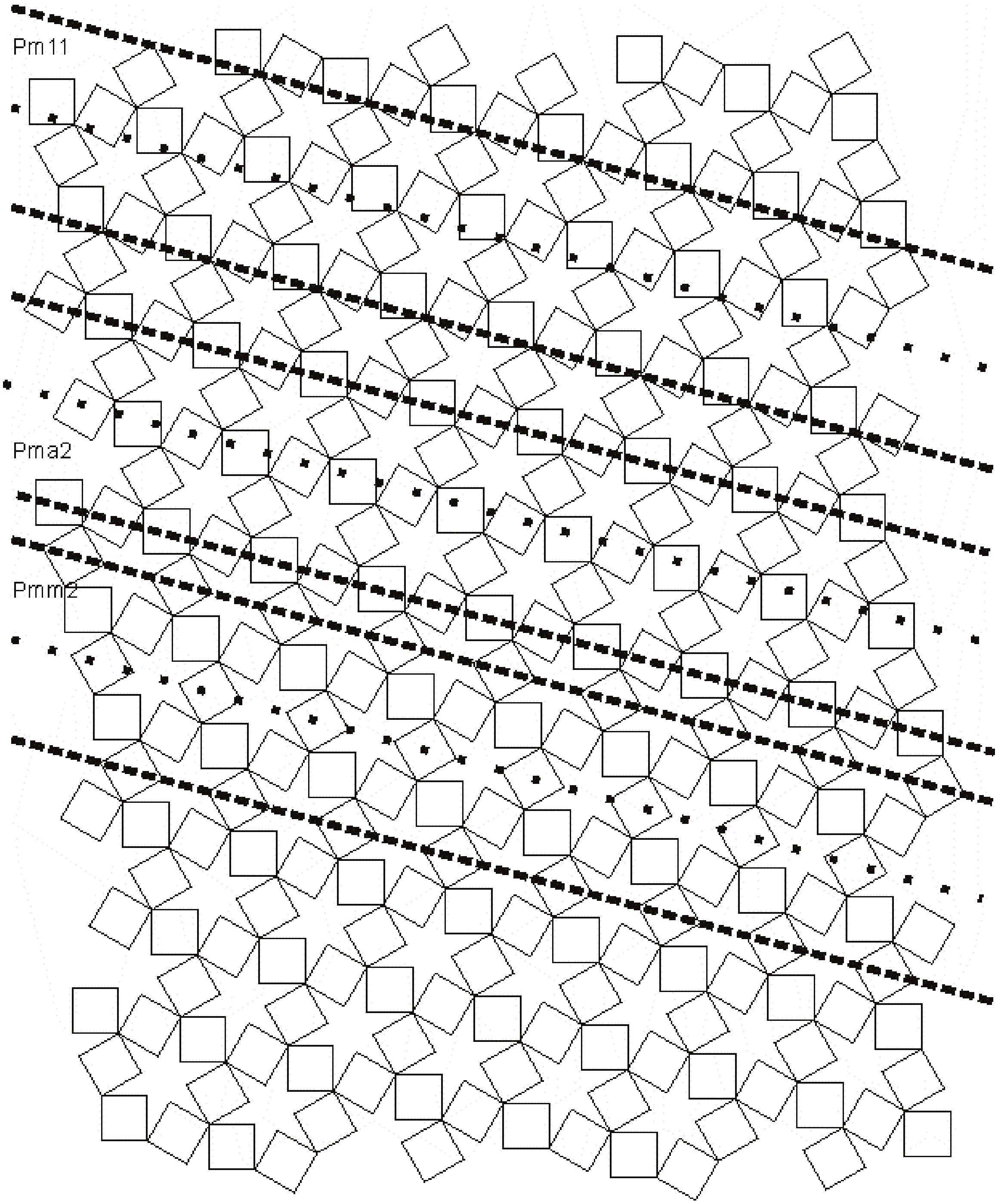


The p111 and p112 patterns will exist in every direction that contains translation, as in the p3 and p4 wallpaper patterns.

The p6m pattern

The p6m pattern contains the same border patterns as the p4m though it has a different structure. Where p4m has four directions of reflection, the p6m has six. Because of this the pmm2 and the pma2 will exist in six directions. The pm11 pattern will also be found in six directions, those that are perpendicular to the axes. Again as in the p4m, the p1m1 and p1a1 will not exist since there exist half turn points on all reflection and glide reflection axes. The five border pattern types that can be found are shown below.

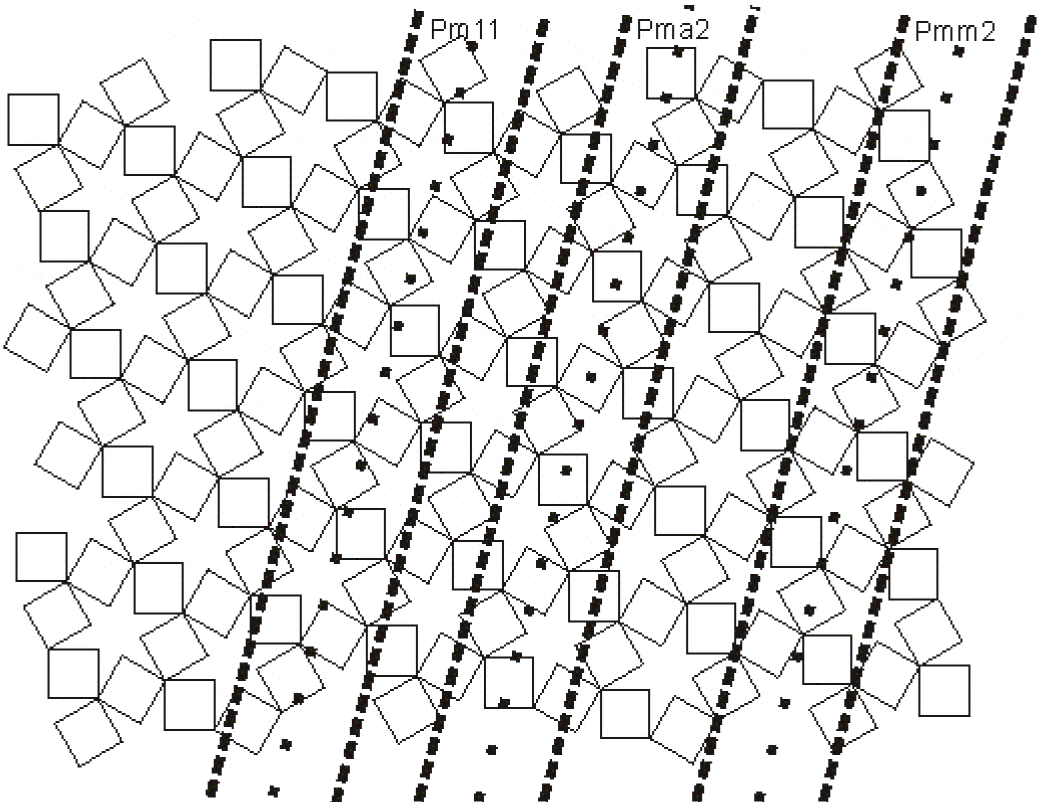
Figure 21a: The border patterns within the p6m that correspond to the p31m



The structure of the p6m is that of the p31m superimposed over the p3m1, in the same way as the cmm and pgg are combined to create the p4g. In effect, it has all the reflection and glide reflection of both types of pattern. Because of this structure, the three patterns pm11, pmm2, and pma2 exist in six different directions, three that correspond to the p3m1 and three that correspond to the p31m.

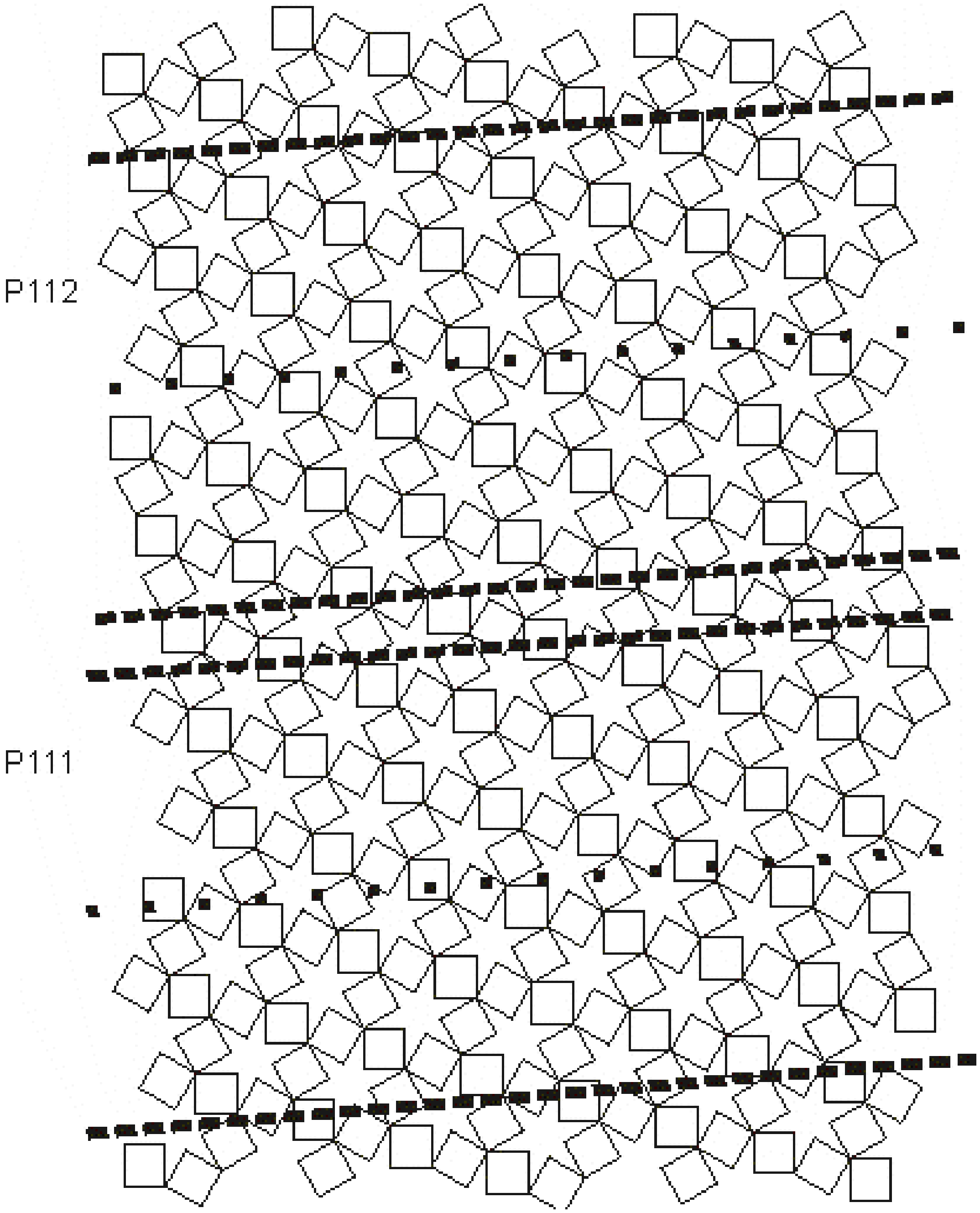
Figure 21a shows patterns that are constructed using the translation vectors from the p31m. Here the pm11 corresponds to the p111, the pma2 to the p1a1, and the pmm2 to the p1m1.

Figure 21b: The border patterns within the p6m that correspond to the p3m1



The border patterns in figure 21b correspond to those found in the p3m1 pattern and thus use a smaller vector to determine width. Note that the translation vector in this figure is equal in length to the width of the border patterns in figure 21a and vice versa.

Figure 21c: The p111 and p112 patterns within the p6m



Finally figure 21c shows the stray, or, as they are shown in the p4m, the slanted patterns, the p111 and p112. Ironically, the p6m, which of all the wallpaper patterns is richest in isometries, presents the greatest challenge when one searches for these two patterns. Because neither pattern can be perpendicular to a reflection axis there are six directions in which these two borders will not exist. Any border pattern found in another direction will be either a p112 or a p111. Though as we have seen before, p111 patterns are more prominent and p112 patterns appear only when by shifting a p111 we bring a center line into a row of rotation points.

Conclusions:

We can sum up all of the cases into a set of conditions that decide whether a certain kind of border pattern will tile some type of wallpaper pattern.

1. Every wallpaper pattern can be tiled by a p111.*

2. A wallpaper pattern can be tiled by a p112 if and only if it has half turn.*

3. A wallpaper pattern can be tiled by a pm11 if and only if it contains reflection axes.

4. A wallpaper pattern can be tiled by a p1a1 if and only if it contains glide reflection axes devoid of rotation points.

5. A wallpaper pattern can be tiled by a p1m1 if and only if it contains reflection axes devoid of rotation points.

6. A wallpaper pattern can be tiled by a pma2 if and only if it contains a glide reflection axis that is perpendicular to some reflection axis.

7. A wallpaper pattern can be tiled by a pmm2 if and only if it contains perpendicular reflection axes.

*Again, these results are dependent upon the existence of perpendicular translation vectors. The vectors that define these two types of border patterns may not both lie perpendicular to reflection axes.

Justification:

1.

All wallpaper patterns have infinitely many directions of translation and none has more than six directions of glide reflection or refection axes. Pick a direction of translation not parallel or perpendicular to any axis. Any border pattern in this direction can only be a p112 or a p111.

For the sake of contradiction, assume that for some wallpaper pattern, all border patterns in this direction are p112 patterns. In effect, any line drawn in this direction will contain infinitely many rotation points. Consider a rectangle with length equal to that of the smallest translation vector in this direction and with width equal to the width of the border pattern drawn in this direction (perpendicular translation assumption). Since this rectangle is made up of perpendicular translation vectors, we can say that every type of rotation point exists inside of it. Since all lines in this direction include rotation points we can say that for every possible line there is a rotation point within the rectangle in question. An infinite amount of possible lines implies an infinite amount of rotation points in a finite space: this is a contradiction, since the rotation points in a wallpaper cannot be arbitrarily close.

Thus some line must pass through no rotation points and will therefore function as a center line of some p111 border pattern, and so we can say that all wallpaper patterns contain some p111.

2.

If we have a single rotation point, in a given wallpaper pattern, then we know that there will be infinitely many translation vectors which take this point to like half turn points in the directions of these vectors. Lines containing rotation points will exist in infinitely many directions. Any border pattern such that one of these lines of half turn points is its center line will be either a p112, a pma2, or a pmm2.

To show that one of these patterns is a p112, we assume for the sake of contradiction, that for some wallpaper pattern with half turn, none of the border patterns centered as described would be a p112. Thus this pattern must have reflection axes in infinitely many directions in order to have only pmm2 and pma2 in infinitely many directions. This is a contradiction since the wallpaper pattern with the most directions is the p6m with only six directions of axes. Therefore we know that any wallpaper pattern that contains half turn points will contain some p112 pattern.

3.

It is clear to see in our patterns that all reflection axes in wallpaper patterns will run parallel to infinitely many like axes. If we consider any border pattern running perpendicular to these axes, it must be a pmm2, a pma2 or a pm11. Since there must be some point on the reflection axis which is not also contained within some perpendicular reflection or glide reflection axis, we can simply pick the pattern with a center line traveling through this point and we are guaranteed to have a pm11.

4.

We have a glide reflection axis devoid of rotation points, and so we can say that it will also have no perpendicular reflection, since this would create half turn centers on the axis in question. Clearly, a border pattern in which this glide reflection is the center line will be a p1a1.

5.

Call the reflection axis in the p1m1 a glide reflection with the zero vector as its translation and examine the proof of number 4.

6.

Consider a wallpaper pattern with glide reflection and rotation points existing on a single line: this is implied by the existence of a reflection axis perpendicular to the glide reflection, as pointed out in number 4. We can find some border pattern within the wallpaper pattern that has the line in question for a center. Clearly this border pattern will contain both glide reflection and rotation, and will thus be a pma2.

7.

Call the reflection in the pmm2 a glide reflection with the zero vector as its translation and examine the proof of number 6.

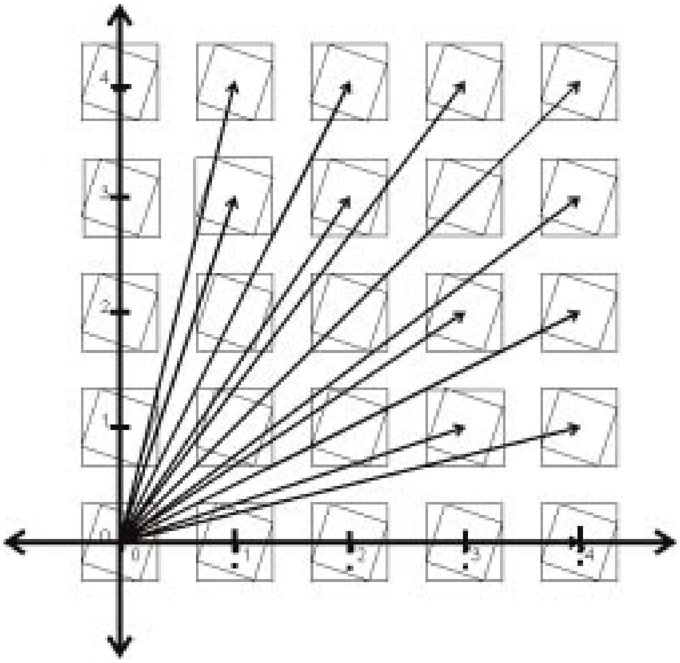
Appendix 1

The Problem:

The characteristics of a border pattern include having constant finite width and infinite length as well as translation. Of course the translation must be parallel to the direction of the border pattern in question. We call a piece of a pattern which has constant finite width and infinite length a strip. The question is then whether a strip chosen from a wallpaper pattern must be a border pattern; in effect must there be a translation vector in every direction.

We have been referring to patterns which can exist in infinitely many directions, but of course they will not exist in all directions. This is due to the existence of translation vectors in infinitely many directions. A simple case to consider is that of the p112 in the p4. The p4 pattern has a grid structure which can be used as a coordinate system if each fourfold point of one type is a point with integer coordinates, as in figure 22a. If we pick a center line which includes the point (0, 0) then any line with a rational slope will be the center line of a p112 border pattern, as shown.

Figure 22a: Possible center lines of a p112 in the p4

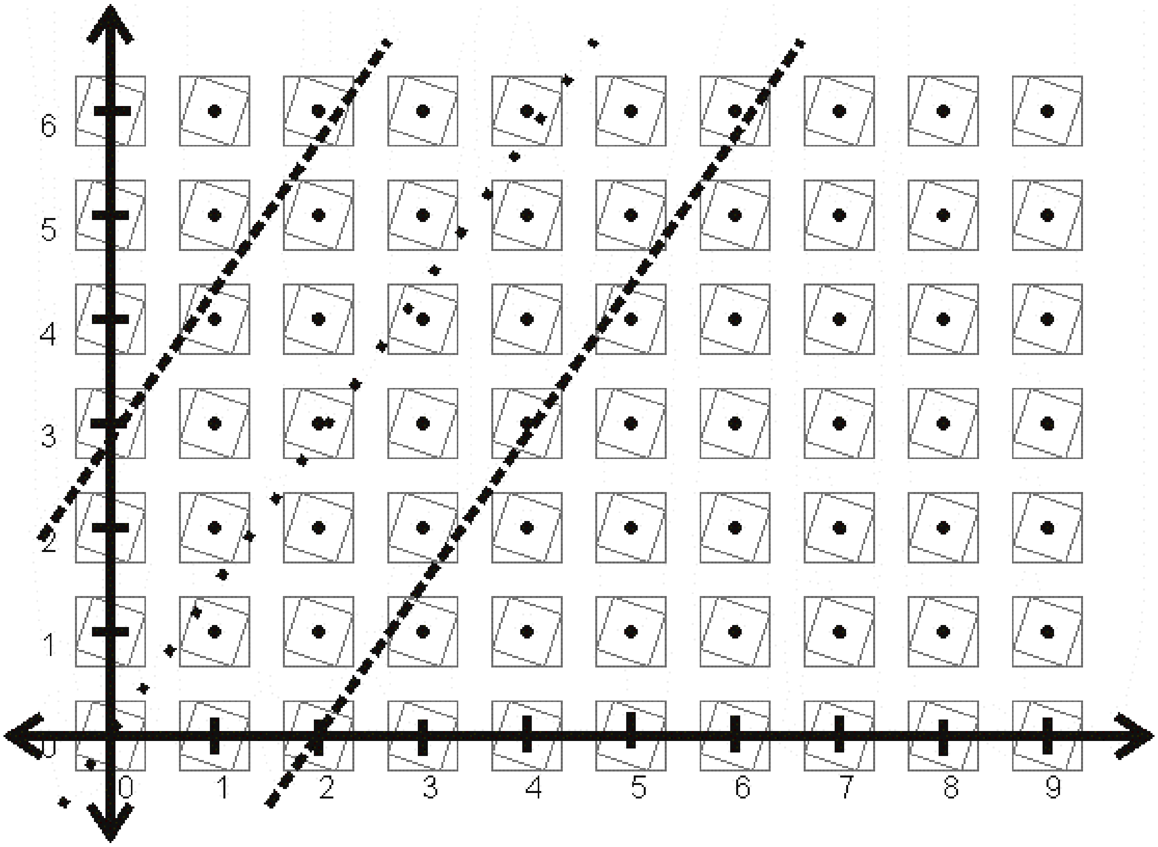


These center lines will clearly belong to a p112 pattern, since they each eventually include an infinite amount of rotation points. I have shown only the few directions which will join two centers within this small segment, but it should be clear that an infinite amount of directions could connect the point (0, 0) with some other point with integer coordinates. Any line from the point (0, 0) with a rational slope m/n will include the rotation point (0, 0) and also the rotation points (n, m), (2n, 2m), (3n, 3m), etc.

Of course each p112 pattern implies a p111 pattern in the same direction, with its center line shifted to avoid rotation points. In the situation of the p112 with slope m/n, a corresponding p111 could have a center line of slope m/n passing through ($2^{1/2}$, 0).

A strip chosen from the p4 wallpaper pattern will not be a border pattern when its slope is irrational. A strip with irrational slope can contain a maximum of one rotation point, since containing two rotation points would imply the existence of two points with integer coordinates which are connected by a line with an irrational slope, which is clearly impossible.

Figure 22b: A strip which is not a border pattern

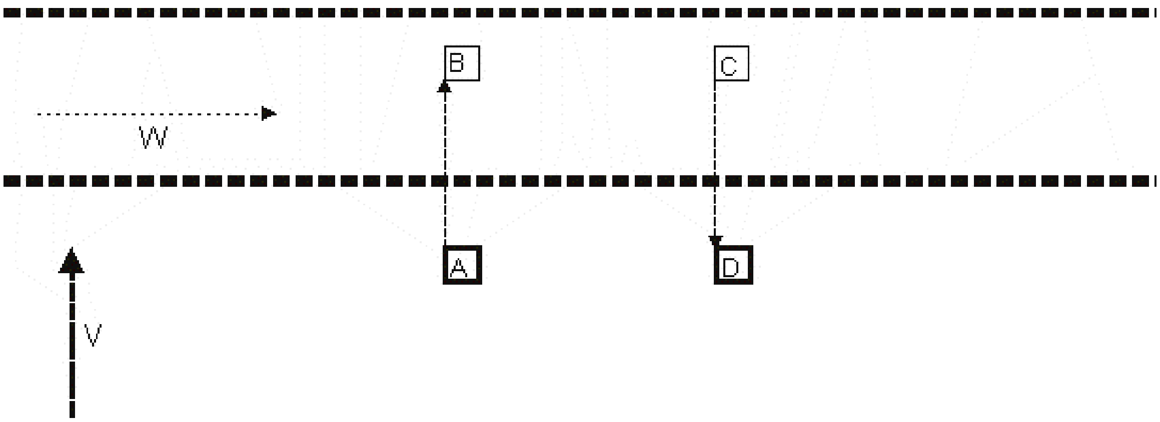


The above example shows a strip with slope $2^{1/2}$, which is irrational. The strip above is thus not a border pattern: its center line will never cross a second rotation point after (0, 0), so it cannot possibly have translation.

Though not all wallpaper patterns have this obvious square grid, all wallpaper patterns with infinitely many pairs of perpendicular translations will contain p111 patterns in infinitely many directions, along with infinitely many directions of strips that are not border patterns. Though the p111 patterns will exist in countably infinitely many directions, there will also be strips that are not border patterns in uncountably infinitely many directions.

Now the question arises of whether there might exist some border pattern in a direction where the wallpaper pattern does not have translation. The proof that such a pattern does not exist will resemble the proof of our original set of four lemmas.

Figure 23: Extending translation from border pattern to tiled wallpaper pattern



Lemma 5: If a border pattern tiles a wallpaper pattern then the border pattern's translation vector will work on the entire wallpaper pattern.

Proof:

Similarly to our earlier lemmas, some integer multiple of the smallest perpendicular translation vector V will take any given point A into the border pattern to some point B. The point B, since it is within the border pattern, will translate, by the border pattern's translation vector W, to the point C. As in lemma 4, since V works in an upwards direction, we know it also works in the downwards direction, and so the point C can be translated by the same integer multiple of V downwards to D. The point D is also the image of A under translation by W, so W must work for the entire wallpaper pattern.

Appendix 2

The Problem:

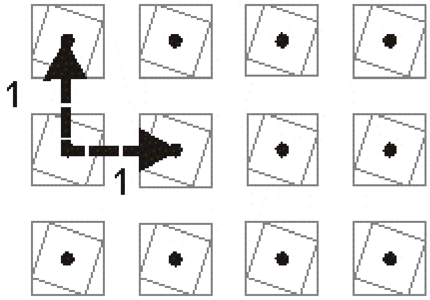
As we have seen the tiling of a certain wallpaper pattern by a p111 or p112 pattern is frequently dependent on the perpendicular translation assumption. Here, I will explain when this assumption is unneeded and when it is invalid.

The assumption that there exist perpendicular translation vectors is unneeded when their existence is guaranteed because of other isometries, namely 60^o, 90^o, and 120^o rotation. Certain wallpaper patterns make the existence of needed perpendicular translation vectors certain, such as the p4, shown in Appendix 1.

Any wallpaper pattern with 90^o or 120^o (or 60^o) rotation will have perpendicular translations in infinitely many pairs of perpendicular directions. To understand this idea, consider the grid structures shown in figures 14b and 17c. For any vector drawn between two rotation points there will be some perpendicular translation vector which will also connect two like points. This endows all threefold, fourfold or sixfold patterns with automatic perpendicular translation. Note that perpendicular translation in sixfold patterns follows from perpendicular translation in threefold patterns since they will share a common grid. We now need only consider the 180^o and 360^o patterns.

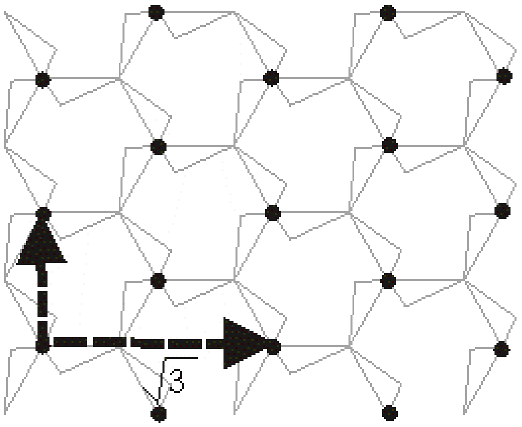
Rotations of 120^o, 90^o, and 60^o will bring about perpendicular translation in infinitely many directions since grids shown in figures 14b (for the case of 90^o) and 17c (for the 60^o and 120^o) each create a set of obvious translation vectors which will allow for infinitely many perpendicular pairs.

Figure 24a: Perpendicular translation caused by fourfold rotation



As stated in Appendix 1, the grid in a p4 or any wallpaper pattern with fourfold rotations will have infinitely many directions of translation vectors since, using the grid shown in figures 22a and 22b, we have translation by any vector represented by integers. It is possible for us to find infinitely many pairs of perpendicular translation vectors. Given some vector (a, b) with a, b being integers, then we can find some (c, d) where c and d are integers and (a, b) is perpendicular to (c, d). In order to find (c, d) we need the dot product, (a, b)**o**(c, d) = 0, and so ac + bd = 0, and a/b = -d/c. Clearly for any a and b there exist some integers c and d for which this is true. For each a, b there will clearly exist some c, d for which this is true. For example, in the case of the p112 in figures 17b and 17c, we may set a = -1, b = 3, c = 3, d = 1.

Figure 24b: Perpendicular translation caused by threefold rotation



Patterns with threefold rotations have more restrictions on the translation vectors which can be found, since one of the shortest pair of perpendicular translation vectors has length $3^{1/2}$ as shown. Because of this, we represent any translation vector as $(a*3^{1/2}, b)$, where a and b are integers. Given a vector $(a*3^{1/2}, b)$, where a and b are integers, we can find some $(c*3^{1/2}, d)$, such that these vectors will be perpendicular. For this we need their dot product to be zero. We want $(a*3^{1/2})(c*3^{1/2}) + bd=0$, and so we need 3a/b = -d/c. For example, in the case of the slanted p111 in figures 14a and 14b, we may, after rotating the vectors in figure 14b by 120^o, set a = 1, b = 2 (short vector), c = -2, d = 3 (long vector).

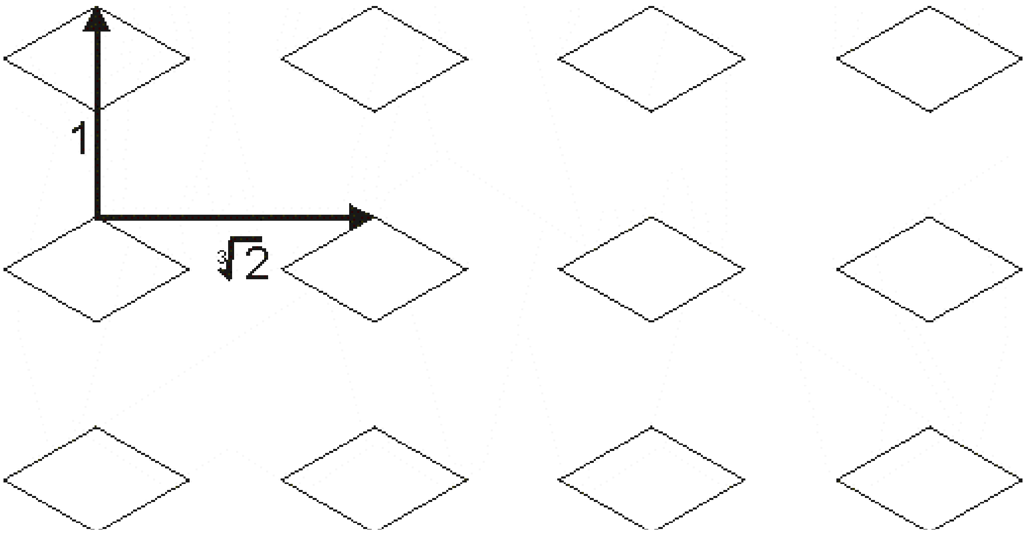
For more examples of the situations in fourfold and threefold pattern examine the border patterns found in these patterns, as well as figures 14b and 17c for examples of various sets of perpendicular translation vectors.

This leaves us with only a few situations to consider. As stated in our chart, the existence of p111 and p112 in the p2, pmm, and cmm as well as the p111 in the p1 dependent on the existence of perpendicular translation. Also, any skewed p111 or p112 patterns in other wallpaper patterns will rely on this. All other cases have been shown to have perpendicular translation vectors because of other isometries.

Situations where there are no skewed directions in which perpendicular translation exists can occur in only these four patterns. As we have seen in the case of p2, pmm and cmm, the existence of p111 implies p112 and vice versa. This is due to the fact that p111 is simply a p112 pattern which has been moved until its center line no longer contains rotation points and a p112 is a p111 whose the center line has been moved onto rotation points. Thus the existence of any skewed direction of translation with another direction of translation perpendicular to it will imply the existence of both p111 and p112 patterns in any wallpaper pattern. Also, the nonexistence of perpendicular skewed translation vectors implies the nonexistence of p111 and p112 in these three wallpaper patterns.

The vectors in the following pmm create a situation with no pair of perpendicular translations in skewed directions. The labels represent the length of each vector.

Figure 25: A pmm with no skewed perpendicular translations



This example has two perpendicular translation vectors of length 1 and $2^{1/3}$. Though these two are perpendicular, they will not lead to any p111 or p112 patterns, since both are parallel to reflection axes. Still, the wallpaper pattern has infinitely many slanted translation vectors which are sums of integer multiples of these two vectors.

In order for two of these slanted vectors to be perpendicular, the dot product of their coordinates would have to be equal to zero. To show that this is impossible, assume the opposite, that there exist two perpendicular translation vectors in this pmm, neither of which is a multiple of the two given translation vectors; then we can call these two slanted vectors $a*(0, 1) + b*(2^{1/3}, 0)$ and $c*(0, 1) + d*(2^{1/3}, 0)$, for some a, b, c, and d which are all nonzero integers. This gives us $(b*2^{1/3}, a)\mathbf{o}(d*2^{1/3}, c) = 0$, and so $bd*2^{2/3} + ac = 0$. Then $ac/bd = -2^{2/3}$. Since a, c, b and d are all nonzero integers, this is a contradiction. So there are no perpendicular translation vectors other than the multiples of the given orthogonal pair.

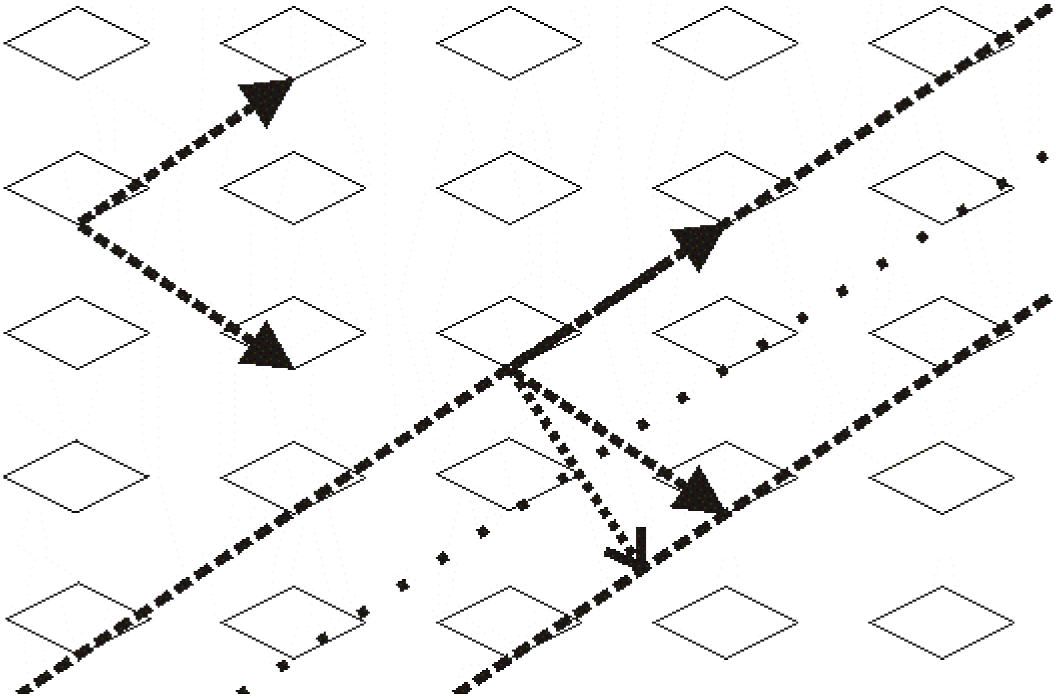
Appendix 3

The Problem:

Does the example in figure 25 mean that some pmm's cannot be tiled by any p111 or p112 patterns?

No, it simply means that our definition must be stretched to accommodate this situation. If a wallpaper pattern has no pair of skewed perpendicular translation vectors, then another set of translation vectors can be used. This pair of vectors will create border patterns copies of which will tile the wallpaper pattern, though in a shifted way. For example, in our pmm example, we have infinitely many skewed translation vectors. Pick any pair of vectors that are not parallel. Use one to define the direction and the component of the other which is perpendicular to the first for the width of the border pattern, as shown below.

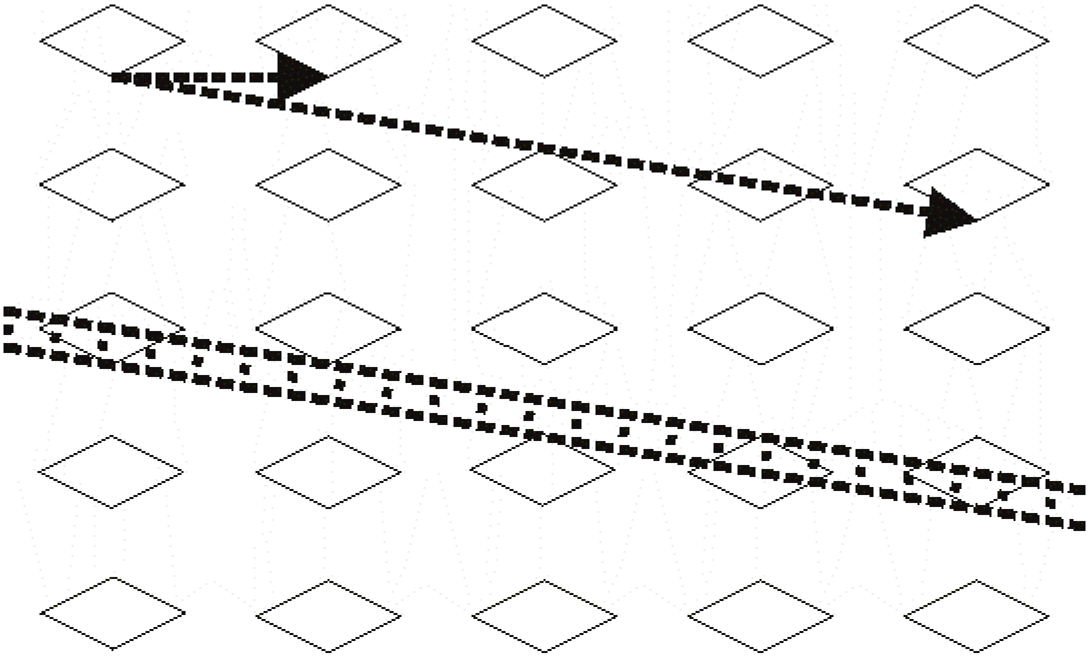
Figure 26a: A border pattern defined by non perpendicular translation vectors



This example shows a pair of translation vectors that are relatively close to perpendicular. Because of this, the resulting p111 border pattern will clearly tile the pmm. The resulting tiling in figure 26a is similar, but by no means identical, to the tiling by the p111 in figure 11b.

Another pair of vectors that creates a p112 that is less obviously going to tile the plain is shown below.

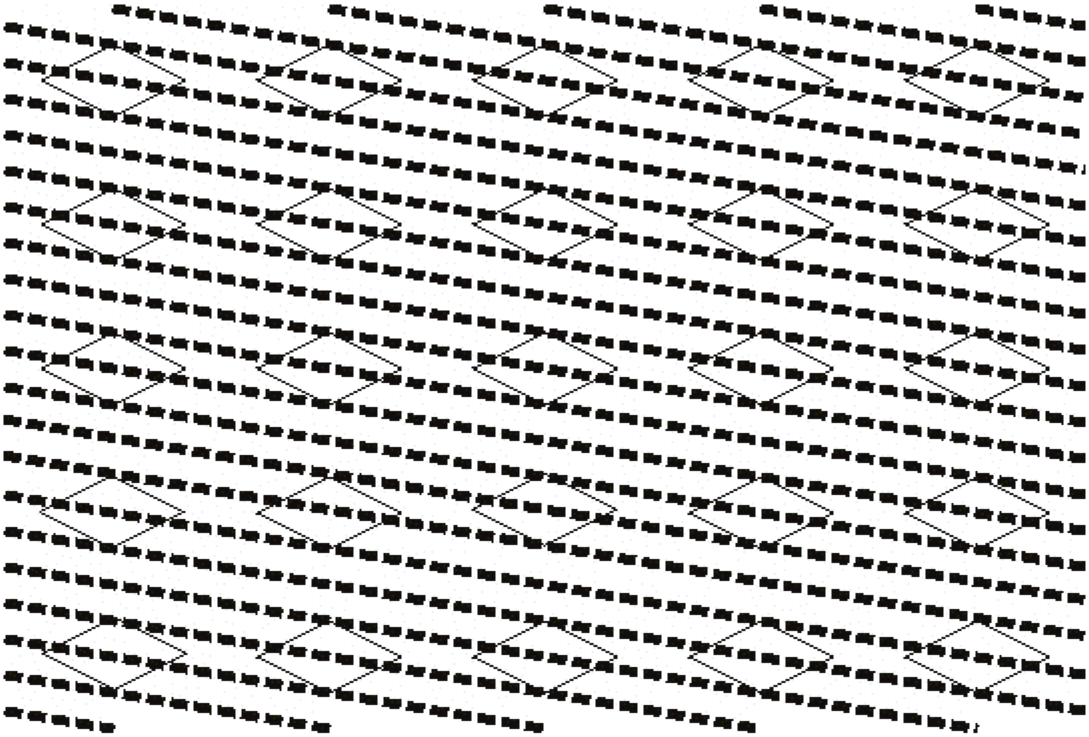
Figure 26b: A second border pattern defined by non perpendicular translation vectors



This pattern is a p112 as opposed to a p111 due to the two kinds of half turn points that it crosses. Moving from left to right it first crosses the rotation point which lies in between four diamonds, and then through a second point which is directly between two diamonds, horizontally.

That this p112 border pattern will tile the pmm is not obvious, but in fact any border pattern created in this way will tile the parent pattern. For example, examine the tiling by the border pattern constructed in figure 26b shown below.

Figure 26c: The tiling of the pmm by the p112 pattern in figure 24b



Looking closely at each border pattern in figure 26c, one can see that they are identical. The patterns are shifted so as to line up correctly. Each pattern is translated by the horizontal translation vector onto a copy of itself.