Article Review of “An Application of Elementary Group Theory to Central Solitaire”

By Arie Bialostocki

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Arie Bialostocki, the author of this article, studied at Tel-Aviv University in Israel. Dr. Bialostocki later became a full-time professor at the University of Idaho. Dr. Bialostocki’s main interest is Ramsey theory, especially zero-sum problems. In this article, Dr. Bialostocki discusses the classical puzzle of central solitaire, also known as peg solitaire; he is able to present to readers insight into a puzzle that may give students a lasting impression of the power of group theory without them even knowing. The main goal of Dr. Bialostocki’s article is to use the structure of the Klein 4-group to analyze the puzzle of central solitaire and leave the reader with an application of abstract mathematics.

Central solitaire is played on a game board consisting of 33 holes that are filled with 32 pegs (board depicted below). The objective of the game is to remove as many pegs as possible. The way you remove pegs is by a horizontal or vertical jump of one peg over only one adjacent piece, which is then removed, and the jumping peg is placed into a vacant hole next to the removed peg. The game is over when there are no more possible moves or only one peg remains.

Dr. Bialostocki explored what he calls the five locations theorem. In this theorem, Dr. Bialostocki used the Klein 4-group, or the additive group $G=\{0, x, y, z\}$, $\oplus$). “The group $G$ obeys the following two properties: i) Every element is its own inverse and ii) the sum of any two distinct nonzero elements is equal to the third nonzero element” (Bialostocki, p. 209). We then use a tessellation, or repetitive pattern, of these three elements of $G$ to cover our board. We define the function $S$, “To be the sum of the
elements of $G$ in the tessellated board (Figure A) that correspond to the locations of the pegs in the given configuration” (Bialostocki, p. 209). For Dr. Bialostocki’s analysis, the key is that value of the signature function $S$ does not change, which automatically happens during the game. For example, if we are left with the board in Figure B, we get the sum, $S$, to be 0. For the game of central solitaire the initial configuration is equal to $y$, therefore throughout the whole game, only configurations with signature $y$ are possible.

Dr. Bialostocki states the theorem; “There are at most five locations in which a single peg can form the final configuration to end the game”, depicted in figure C to the left (Bialostocki, p. 210). In summary, his proof explains that the signature function $S$ must equal $y$, there are only \( \frac{33}{3} = 11 \) possible holes (all holes marked with $y$) in which the last peg can be left. Since this board is symmetric we must take into consideration the eight symmetries that can be applied to the board. Only holes that remain marked $y$ through these symmetries are possible ending holes for a single peg. This eliminates 6 of the possible 11 locations for a single peg; therefore there are only five remaining locations (depicted in Figure C). Dr. Bialostocki proved in this article that it is impossible to leave a single peg in any location other than the five depicted above in Figure C. As an added bonus, Dr. Bialostocki included a simple algorithm for leaving one central peg.

I found it very interesting for Dr. Bialostocki to place the group $G$ on top of the board making the function $S$ easier to understand and being able to place value on the peg holes and configurations. In his article, Dr. Bialostocki poses two extra problems, and
Problem 2 reads: “A common version of peg solitaire has 14 pegs in 15 hole arrayed in an equilateral triangular array, with one corner hole empty. Using the technique above, find ten locations in which it is impossible to leave a single peg. Can a single peg be left in each of the other 5 locations?” (Bialostocki, p. 210). In response to his article I decided to look further into this question as I was intrigued. The figure to the left (Figure D) is the tessellated board using the same tessellation as we did for the 33 peg solitaire board. There are three different cases where we can have a corner hole empty to start. Case #1 we remove the top x. We are left with 5 possible ending locations, the holes labeled with an x. Since the only symmetry of the board is reflection through the altitude from the vacant vertex, we don’t have to eliminate any of the locations. For Case #2, we remove the left corner, labeled with y. We are again left with 5 possible ending locations and none are eliminated due to problems with the symmetry. Case #3, we remove the right corner, labeled with z. Again, we are left with 5 possible ending locations and none are eliminated due to problems with symmetry.

In each of these cases there are 10 locations that it is impossible to leave a single peg, but we notice each of these cases is just a simple rotation. Through the application of the technique Arie Bialostocki used, we get that it is impossible to leave a peg in 10 locations; if we remove the x corner, it is impossible to end on any y, z spot, and vice versa for the other two corners.

This article was a good read for someone with my experience of group theory and abstract algebra. I was able to understand the majority of it through reading and working out the examples. I am interested in group theory and interested in working out the
problems that the article left the reader with. Other math majors should read this article to see an example of an application of group theory translated to a simple everyday game. It allows the reader to understand some group theory without even knowing it. The language of the article allows for easy comprehension and is a definite recommendation for anyone interested in mathematics.
